1. Are the following series convergent? Why?

   (i). $\sum_{n=1}^{\infty} e^{in}$

   Solution: $|e^{in}| = 1$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} e^{in} \neq 0$, which implies the series is divergent.

   (ii). $\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{i}{n}$

   Solution: The imaginary part of the series is the series $\sum_{n=1}^{\infty} -\frac{1}{n}$, which is divergent, so the series is divergent.

   (iii). $\sum_{n=1}^{\infty} \frac{e^{in}}{n^2}$

   Solution: $\sum_{n=1}^{\infty} |\frac{e^{in}}{n^2}| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so the series is absolutely convergent, hence convergent.

2. If $\lim_{n\to\infty} z_n = z$, prove $\lim_{n\to\infty} |z_n| = |z|$.

   Solution: For any $\epsilon > 0$, since $\lim_{n\to\infty} z_n = z$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|z_n - z| < \epsilon$. Then by triangle inequality, $n > N$ implies $||z_n| - |z|| \leq |z_n - z| < \epsilon$, so $\lim_{n\to\infty} |z_n| = |z|$

3. If $\sum_{n=1}^{\infty} z_n = z$, prove $\sum_{n=1}^{\infty} \bar{z}_n = \bar{z}$

   Solution: For any $\epsilon > 0$, since $\sum_{n=1}^{\infty} z_n = z$, there exists $N_\epsilon \in \mathbb{N}$ such that $N > N_\epsilon$ implies $|z - \sum_{n=1}^{N} z_n| < \epsilon$, which further implies $|\overline{z - \sum_{n=1}^{N} z_n}| < \epsilon$,.i.e. $|\bar{z} - \sum_{n=1}^{N} \bar{z}_n| < \epsilon$, so $\sum_{n=1}^{\infty} \bar{z}_n = \bar{z}$

4. Find the Taylor series expansion of $f(z) = \frac{1}{z}$ at $z_0 = 2$.

   Solution:

   $f(z) = \frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2}\frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{z-2}{2})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-2)^n$

   $(|z-2| < 2)$

5. Find the Taylor series expansion of $f(z) = \frac{z}{z^4+4}$ at $z_0 = 0$.

   Solution:

   $\frac{z}{z^4+4} = \frac{z}{4}\frac{1}{1+\frac{z^4}{4}} = \frac{z}{4}\sum_{n=0}^{\infty}(-\frac{z^4}{4})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} z^{4n+1}$

6. Consider $f(z) = \log z$ in the branch $a < \theta < a + 2\pi$, where a is a constant real number such that $a \notin 2\pi\mathbb{Z}$. Find the Taylor series expansion of $f(z)$ at $z_0 = 1$.

   Solution: $f(z) = \log z$, so $f^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{z^n}$, $f^{(n)}(1) = (-1)^{n-1}(n-1)!$ so $\log z = \log 1 + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \log 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$
   (If a is in Quadrant I or IV, $|z-1| < |\sin a|$; otherwise $|z-1| < 1$)