1. C is the circle $|z| = 2$, positively oriented. Compute

$$\int_C \frac{1}{z^2+1}\,dz$$

Solution: Let C_1 be the circle $|z+i| = \frac{1}{2}$, C_2 be the circle $|z-i| = \frac{1}{2}$, both positively oriented. The function $f(z)$ is analytic on and in between C, C_1, C_2, so

$$\begin{aligned}
&\int_C \frac{1}{z^2+1}\,dz\\
=&\frac{1}{2i}\Big(\int_C \frac{1}{z-i}\,dz - \int_C \frac{1}{z+i}\,dz\Big)\\
=&\frac{1}{2i}\Big(\int_{C_1} \frac{1}{z-i}\,dz + \int_{C_2} \frac{1}{z-i}\,dz - \int_{C_1} \frac{1}{z+i}\,dz - \int_{C_2} \frac{1}{z+i}\,dz\Big)\\
=&\frac{1}{2i}(0 + 2\pi i - 2\pi i - 0)\\
=&0
\end{aligned}$$

2. $f(z) = u(x,y) + iv(x,y)$ is entire. If there exists $u_0 \in \mathbb{R}$ such that $u(x,y) \le u_0$ for all x, y, prove $u(x,y)$ is a constant function. [Hint: consider $e^{f(z)}$]

Solution: Since $f(z)$ is entire, $e^{f(z)}$ is also entire.

$|e^{f(z)}| = |e^{u(x,y)+iv(x,y)}| = e^{u(x,y)} \le e^{u_0}$, so $e^{f(z)}$ is bounded, by Liouville's Theorem, we get $e^{f(z)} \equiv C$ for some constant C.

We therefore have $f(z) \in \{LogC + 2\pi ki \in \mathbb{C} | k \in \mathbb{Z}\}$. Note $f(z)$ is continuous function on the connected domain $\mathbb{C}$, and the set $\{LogC + 2\pi ki \in \mathbb{C} | k \in \mathbb{Z}\}$ is discrete, so $f(z)$ has to be a constant function.

3. $p(z) \in \mathbb{C}[z]$ is a non-constant polynomial. c is a root of $p(z)$. Prove:

(i). $z^k - c^k = (z-c)(z^{k-1} + z^{k-2}c + ... + zc^{k-2} + c^{k-1})$

Solution:

$$
\begin{aligned}
&(z-c)(z^{k-1}+z^{k-2}c+...+zc^{k-2}+c^{k-1})\\
=&(z-c)z^{k-1}+(z-c)z^{k-2}c+...+(z-c)c^{k-1}\\
=&z^k-z^{k-1}c+z^{k-1}c-z^{k-2}c^2+...+zc^{k-1}-c^k\\
=&z^k-c^k
\end{aligned}
$$

(ii). Making use of (i) to prove there exists a polynomial $q(z)$ such that

$$p(z) = p(z) - p(c) = (z-c)q(z)$$

Solution:

Let $p(z) = a_n z^n + ...a_1 z + a_0$. ($n > 0$, $a_n \neq 0$)

$$
\begin{aligned}
p(z) =&p(z)-p(c)\\
=&a_n(z^n-c^n)+...+a_1(z-c)\\
=&a_n(z-c)(\sum_{i=0}^{n-1} z^{n-1-i}c^i)+...+a_1(z-c)\\
=&(z-c)(\sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} z^{k-1-i}c^i)
\end{aligned}
$$

4. (i). C is the circle $|z| = 1$. Prove $g(z) = \frac{z-1}{z+1}$ maps $U = \{x+iy \in \mathbb{C} | x > 0\}$ to the interior of C.

   Solution: If $z = x + iy$ with $x > 0$, then

   $$|z-i| = |x-1+iy| = \sqrt{(x-1)^2+y^2} < \sqrt{(x+1)^2+y^2} = |x+1+iy| = |z+1|$$

   So $|g(z)| = |\frac{z-1}{z+1}| = \frac{|z-1|}{|z+1|} < 1$, we see $g(z)$ is inside C.

   (ii). f is an entire function. L is a straight line on $\mathbb{C}$. If the image of f all lie on the same side of L, prove f is a constant function.

Solution: We can find a translation T and a rotation R on $\mathbb{C}$ such that $R \circ T$ sends L to the y-axis and the image of $R \circ T \circ f$ is to the right of $y - axis$.

Consider $g \circ R \circ T \circ f$, which is entire and by (i), its image is inside C, so it is bounded. We conclude $g \circ R \circ T \circ f$ is constant, and g, R, T are all injective, we conclude f is constant.

(iii). Prove Question (2) again using Question 4(ii).

Solution: If $u(x, y) \leq u_0$, then the image of f all lies to the left of the straight line $x = u_0$, so by (ii), f is a constant.