1. Prove $\int_C Log(z+2)\, dz = 0$, where C is the positively oriented unit circle $|z| = 1$

   Solution: The branch cut for $Log(u)$ is the nonpositive real axis, so the image of $|z| \leq 1$ under $u = z + 2$ is within the domain of the branch of $Log(u)$, the composition function $Log(z + 2)$ is hence analytic on and interior to $|z| < 1$. By Cauchy-Coursat Theorem, the integral $\int_C Log(z + 2)\, dz = 0$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. Prove
$$\int_{C_1} \frac{z}{1 - e^z}\, dz = \int_{C_2} \frac{z}{1 - e^z}\, dz$$

   Solution: $\frac{z}{1-e^z}$ is analytic at z as long as the denominator is not zero. $1-e^z = 0$ if and only if $z = 2\pi k i$, $k \in \mathbb{Z}$, and none of them is in the region between C_1 and C_2, so $\frac{z}{1-e^z}$ is analytic on and between C_1 and C_2, we get
$$\int_{C_1} \frac{z}{1 - e^z}\, dz = \int_{C_2} \frac{z}{1 - e^z}\, dz$$

3. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C is
$$\frac{1}{2i} \int_C \bar{z}\, dz$$

   Solution: Let D be the region enclosed by C.

$$\begin{aligned}
\frac{1}{2i} \int_C \bar{z}\, dz &= \frac{1}{2i} \int_C (x - iy)(\, dx + i\, dy) \\
&= \frac{1}{2i}(\int_C x\, dx + y\, dy) + \frac{i}{2i}(\int_C x\, dy - y\, dx) \\
&= \frac{1}{2i} \iint_D \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\, dA + \frac{1}{2} \iint_D \frac{\partial x}{\partial x} - \frac{\partial (-y)}{\partial y}\, dA \\
&= \frac{1}{2} \iint_D 2\, dA \\
&= \iint_D 1\, dA
\end{aligned}$$

4. Let C be the circle $|z| = 3$ positively oriented. If the function $g(z)$ ($|z| \neq 3$) is defined to be

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds$$

Compute $g(2)$ and $g(4)$.

Solution:

By Cauchy Integral Formula, $g(2) = \int_C \frac{2s^2-s-2}{s-2}\, ds = 2\pi i(2 \times 2^2 - 2 - 2) = 8\pi i$

$g(4) = g(z) = \int_C \frac{2s^2-s-2}{s-4}\, ds$, and $\frac{2s^2-s-2}{s-4}$ is analytic on and inside $|s| = 3$, so by Cauchy-Goursat, $g(4) = 0$.

5. Let C be the unit circle $z = e^{i\theta}$, $-\pi \leq \theta \leq \pi$.

(i). Show that for any real constant a, $\int_C \frac{e^{az}}{z}\, dz = 2\pi i$

Solution:

By Cauchy Integral Formula, $\int_C \frac{e^{az}}{z}\, dz = \int_C \frac{e^{az}}{z-0}\, dz = 2\pi i e^{a \times 0} = 2\pi i$

(ii). Write the above integral in terms of θ to prove

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi$$

Solution:

$$\begin{aligned} 2\pi i = \int_C \frac{e^{az}}{z}\, dz &= \int_0^{2\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} (e^{i\theta})'\, d\theta \\ &= i \int_0^{2\pi} e^{a\cos\theta + ia\sin\theta}\, d\theta \\ &= i \int_0^{2\pi} e^{a\cos\theta} cos(a\sin\theta)\, d\theta - \int_0^{2\pi} e^{a\cos\theta} \sin(a\sin\theta)\, d\theta\, dz \end{aligned}$$

So $\int_0^{2\pi} e^{a\cos\theta} cos(a\sin\theta)\, d\theta = 2\pi$. Let $f(\theta) = e^{a\cos\theta} cos(a\sin\theta)$, note $f(\theta) = f(2\pi - \theta)$, which implies

$$\int_0^\pi e^{a\cos\theta} cos(a\sin\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} e^{a\cos\theta} cos(a\sin\theta)\, d\theta = \pi$$

6. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = az$, where a is a complex constant.

   Solution: For any $z \in \mathbb{C}$, let C_R be the circle centred at z with radius $R > 0$. If $s \in C_R$, then $|s - z| = R$ and $|s| \leq |s - z| + |z| = R + |z|$, so $|f(s)| \leq A|s| \leq A(R + |z|)$.

   Cauchy Integral Formula tells us:

$$\begin{aligned}
|f''(z)| &= |\frac{2!}{2\pi i} \int_{C_R} \frac{f(s)}{(s-z)^3}\, dz| \\
&= \frac{1}{\pi} | \int_{C_R} \frac{f(s)}{(s-z)^3}\, dz| \\
&\leq \frac{1}{\pi} \frac{A(R + |z|)}{R^3} \times 2\pi R \\
&= \frac{A(R + |z|)}{R^2}
\end{aligned}$$

   We see as $R \to +\infty$, $\frac{A(R+|z|)}{R^2} \to 0$, so we conclude $|f''(z)| = 0$, i.e. $f''(z) = 0$ for any $z \in \mathbb{C}$. This implies $f'(z) = a$ for some $a \in \mathbb{C}$, so $f(z) = az + b$ for some $a, b \in \mathbb{C}$. But $|b| = |f(0)| \leq A \times |0| = 0$, we get $b = 0$, $f(z) = az$.