1. Prove $\sin(2z) = 2\sin z \cos z$ for any $z \in \mathbb{C}$.

   Solution:

$$\begin{aligned}
2\sin z \cos z &= 2 \times \frac{e^{zi} + e^{-zi}}{2} \times \frac{e^{zi} - e^{-zi}}{2i} \\
&= \frac{(e^{zi})^2 - (e^{zi})^2}{2i} \\
&= \frac{e^{2zi} - e^{2zi}}{2i} \\
&= \sin(2z)
\end{aligned}$$

2. Show that $\overline{\cos z} = \cos \overline{z}$ for any $z \in \mathbb{C}$.

   Solution:

$$\begin{aligned}
\overline{\cos z} &= \overline{\frac{e^{zi} + e^{-zi}}{2}} \\
&= \frac{\overline{e^{zi}} + \overline{e^{-zi}}}{2} \\
&= \frac{\overline{e^{-y+xi}} + \overline{e^{y-xi}}}{2} \\
&= \frac{e^{-y-xi} + e^{y+xi}}{2} \\
&= \frac{e^{-(x-yi)i} + e^{(x-yi)i}}{2} \\
&= \frac{e^{-\bar{z}i} + e^{\bar{z}i}}{2} \\
&= \cos \bar{z}
\end{aligned}$$

3. Evaluate the integral $\int_0^1 (1+it)^2\, dt$

Solution:

$$\begin{aligned}\int_0^1 (1+it)^2\,dt &= \int_0^1 1-t^2+2ti\,dt\\ &= (\int_0^1 1-t^2\,dt)+i(\int_0^1 2t\,dt)\\ &= \frac{2}{3}+i\end{aligned}$$

4. If $k \in \mathbb{Z}$, evaluate $\int_0^{2\pi} e^{ikt}\,dt$

   Solution:

$$\begin{aligned}\int_0^{2\pi} e^{ikt}\,dt &= \int_0^{2\pi} \cos kt + i\sin kt\,dt\\ &= (\int_0^{2\pi}\cos kt\,dt)+i(\int_0^{2\pi}\sin kt\,dt)\\ &= \begin{cases}2\pi, \text{if } k=0\\ 0, \text{if } k\neq 0\end{cases}\end{aligned}$$

5. If $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a differentiable function and $z(t):\mathbb{R}\longrightarrow\mathbb{C}$ is also a differentiable function, prove $\frac{d}{dt}(z\circ f)=z'(f(t))f'(t)$

   Solution: Write $z(t)=u(t)+iv(t)$. Then $z\circ f(t)=u(f(t))+iv(f(t))$.

$$\begin{aligned}\frac{d}{dt}(z\circ f) &= \frac{d}{dt}u(f(t))+i\frac{d}{dt}v(f(t))\\ &= u'(f(t))f'(t)+iv'(f(t))f'(t)\\ &= (u'(f(t))+iv'(f(t)))f'(t)\\ &= z'(f(t))f'(t)\end{aligned}$$

6. Prove the arclength of a curve on $\mathbb{C}$ is independent of the parametrization, i.e. If $z(t) = x(t) + y(t)i, a \leq t \leq b$ is a differentiable curve, and $t = \phi(\tau) : [c, d] \longrightarrow [a, b]$ is a differetiable bijective function, then the arclength of $z(t), a \leq t \leq b$ equals to the arclength of $z(\phi(\tau)), c \leq \tau \leq d$.

   Solution:

$$
\begin{aligned}
& \int_c^d |\frac{d}{d\tau} z(\phi(\tau))| \, d\tau \\
= & \int_c^d \sqrt{(\frac{d}{d\tau} x(\phi(\tau)))^2 + (\frac{d}{d\tau} y(\phi(\tau)))^2} \, d\tau \\
= & \int_c^d \sqrt{(\frac{dx}{dt}(\phi(\tau))\frac{d\phi}{d\tau})^2 + (\frac{dy}{dt}(\phi(\tau))\frac{d\phi}{d\tau})^2} \, d\tau \\
= & \int_c^d \sqrt{(\frac{dx}{dt}(\phi(\tau)))^2 + (\frac{dy}{dt}(\phi(\tau)))^2} \frac{d\phi}{d\tau} \, d\tau \\
= & \int_a^b \sqrt{\frac{dx}{dt}(t)^2 + \frac{dy}{dt}(t)^2} \, dt \\
= & \int_a^b |\frac{dz}{dt}(t)| \, dt
\end{aligned}
$$