1. Prove $\sin(2z) = 2 \sin z \cos z$ for any $z \in \mathbb{C}$.

Solution:

\[
2 \sin z \cos z = 2 \times \frac{e^{zi} + e^{-zi}}{2} \times \frac{e^{zi} - e^{-zi}}{2i} = \frac{(e^{zi})^2 - (e^{-zi})^2}{2i} = \frac{e^{2zi} - e^{-2zi}}{2i} = \sin(2z)
\]

2. Show that $\overline{\cos z} = \cos \overline{z}$ for any $z \in \mathbb{C}$.

Solution:

\[
\cos \overline{z} = \frac{e^{\overline{z}i} + e^{-\overline{z}i}}{2} = \frac{e^{zi} + e^{-zi}}{2} = \frac{e^{-y+xi} + e^{y-xi}}{2} = \frac{e^{-y-xi} + e^{y+xi}}{2} = \frac{e^{-(x-y)i} + e^{(x-y)i}}{2} = \frac{e^{-\overline{z}i} + e^{\overline{z}i}}{2} = \cos \overline{z}
\]

3. Evaluate the integral $\int_0^1 (1 + it)^2 \, dt$
Solution:

\[
\int_0^1 (1 + it)^2 \, dt = \int_0^1 1 - t^2 + 2ti \, dt \\
= \left( \int_0^1 1 - t^2 \, dt \right) + i\left( \int_0^1 2t \, dt \right) \\
= \frac{2}{3} + i
\]

4. If \( k \in \mathbb{Z} \), evaluate \( \int_0^{2\pi} e^{ikt} \, dt \)

Solution:

\[
\int_0^{2\pi} e^{ikt} \, dt = \int_0^{2\pi} \cos kt + i\sin kt \, dt \\
= \left( \int_0^{2\pi} \cos kt \, dt \right) + i\left( \int_0^{2\pi} \sin kt \, dt \right) \\
= \begin{cases} 
2\pi, & \text{if } k = 0 \\
0, & \text{if } k \neq 0 
\end{cases}
\]

5. If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable function and \( z(t) : \mathbb{R} \rightarrow \mathbb{C} \) is also a differentiable function, prove \( \frac{d}{dt}(z \circ f) = z'(f(t))f'(t) \)

Solution: Write \( z(t) = u(t) + iv(t) \). Then \( z \circ f(t) = u(f(t)) + iv(f(t)) \).

\[
\frac{d}{dt}(z \circ f) = \frac{d}{dt}u(f(t)) + i\frac{d}{dt}v(f(t)) \\
= u'(f(t))f'(t) + iv'(f(t))f'(t) \\
= (u'(f(t)) + iv'(f(t)))f'(t) \\
= z'(f(t))f'(t)
\]
6. Prove the arclength of a curve on $\mathbb{C}$ is independent of the parametrization, i.e. If $z(t) = x(t) + y(t)i, a \leq t \leq b$ is a differentiable curve, and $t = \phi(\tau) : [c, d] \rightarrow [a, b]$ is a differentiable bijective function, then the arclength of $z(t), a \leq t \leq b$ equals to the arclength of $z(\phi(\tau)), c \leq \tau \leq d$.

Solution:

$$\int_{c}^{d} \left| \frac{d}{d\tau}z(\phi(\tau)) \right| d\tau$$

$$= \int_{c}^{d} \sqrt{\left( \frac{d}{d\tau} x(\phi(\tau)) \right)^2 + \left( \frac{d}{d\tau} y(\phi(\tau)) \right)^2} d\tau$$

$$= \int_{c}^{d} \sqrt{\left( \frac{dx}{dt}(\phi(\tau)) \frac{d\phi}{d\tau} \right)^2 + \left( \frac{dy}{dt}(\phi(\tau)) \frac{d\phi}{d\tau} \right)^2} d\tau$$

$$= \int_{c}^{d} \sqrt{\frac{dx}{dt}(\phi(\tau))^2 + \frac{dy}{dt}(\phi(\tau))^2} \frac{d\phi}{d\tau} d\tau$$

$$= \int_{a}^{b} \sqrt{\frac{dx}{dt}(t)^2 + \frac{dy}{dt}(t)^2} dt$$

$$= \int_{a}^{b} \left| \frac{dz}{dt}(t) \right| dt$$