1. C is the contour $z(t) = e^{it}$, $0 \le t \le \frac{\pi}{2}$. $f(z) = z^i$ takes the principal branch $-\pi < \theta < \pi$. Compute

$$\int_C f(z)\,dz$$

Soluion:

$$\begin{aligned}\int_C f(z)\,dz &= \int_0^{\frac{\pi}{2}} e^{i\log(e^{it})}(e^{it})'\,dt \\ &= \int_0^{\frac{\pi}{2}} e^{i(0+it)} ie^{it}\,dt \\ &= \int_0^{\frac{\pi}{2}} ie^{(-1+i)t}\,dt \\ &= \frac{1-i}{2}(-1+e^{-\frac{\pi}{2}}i)\end{aligned}$$

2. Let C be the circle centered at z_0 with radius R, and use the parametrization $z = z_0 + Re^{i\theta}$, $-\pi \le \theta \le \pi$ to compute

$$\int_C (z-z_0)^k\,dz$$

where $k \in \mathbb{Z}$.

Soluion:

$$\begin{aligned}\int_C (z-z_0)^k\,dz &= \int_{-\pi}^{\pi} (Re^{i\theta})^k (z_0 + Re^{i\theta})'\,dz \\ &= R^{k+1} i \int_{-\pi}^{\pi} e^{i(k+1)\theta}\,d\theta \\ &= \begin{cases} 0, k \ne -1 \\ 2\pi i, k = -1 \end{cases}\end{aligned}$$

3. Show that
$$|\int_C \frac{z+4}{z^3-1}\,dz| \le \frac{6\pi}{7}$$
where C is the arc of the circle $|z| = 2$ from 2 to $2i$.

   Soluion: If $z \in C$, we see

   $|z+4| \le |z| + 4 = 2 + 4 = 6$, $|z^3 - 1| \ge |z|^3 - 1 = 2^3 - 1 = 7$

   and the length of the arc C is $2\pi \times 2 \times \frac{1}{4} = \pi$.

   So
$$|\int_C \frac{z+4}{z^3-1}\,dz| \le \frac{6}{7} \times \pi = \frac{6\pi}{7}$$

4. Let C denote the line segment from i to 1. Show that
$$|\int_C \frac{1}{z^4}\,dz| \le 4\sqrt{2}$$

   Soluion: If z is on that lone segment, by geometry it is clear that $|z| \ge \frac{1}{\sqrt{2}}$, so $|\frac{1}{z^4}| \le (\sqrt{2}4^4) = 4$.

   The length of the segment is $\sqrt{2}$, we conclude
$$|\int_C \frac{1}{z^4}\,dz| \le 4 \times \sqrt{2} = 4\sqrt{2}$$

5. Compute $\int_C z^2\,dz$, where C is any contour from i to 3.

   Soluion: z^2 has antiderivative $\frac{z^3}{3}$, so
$$\int_C z^2\,dz = \frac{3^3}{3} - \frac{i^3}{3} = \frac{3^3 - i^3}{3} = \frac{27+i}{3}$$

6. Evaluate the integral
$$\int_C e^z\,dz$$
where C is a closed contour along the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in counterclockwise direction.

   Soluion: e^z has antiderivative e^z, so its contour integral under a closed contour is 0.