1. Prove $|e^{z^2}| \leq e^{|z|^2}$ for any $z \in \mathbb{C}$.

   Solution:

   $|e^{z^2}| = |e^{x^2-y^2+2xyi}| = e^{x^2-y^2} \leq e^{x^2+y^2} = e^{|z|^2}$

2. Compute the following:

   (i). $Log(-ei)$

   Solution: $Log(-ei) = \ln|-ei| + iArg(-ei) = 1 - \frac{\pi}{2}i$

   (ii). $Log(1-i)$

   Solution: $Log(1-i) = \ln|1-i| + iArg(1-i) = \ln\sqrt{2} - \frac{\pi}{4}i = \frac{\ln 2}{2} - \frac{\pi}{4}i$

   (iii). $\log 1$

   Solution: $\log 1 = \ln 1 + iarg(1) = 0 + 2\pi k i$, $k \in \mathbb{Z}$

3. Show that $\log(i^2) \neq 2\log(i)$ on the branch $\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$

   Solution:

   $\log i^2 = \log(-1) = \ln|-1| + iarg(-1) = \pi i$

   $2\log i = 2(\ln|i| + iarg(i)) = 2(\frac{5\pi}{2}) = 5\pi i$

   So $\log i^2 \neq 2\log i$ on this branch.

4. Proving $\log(\frac{1}{z}) = -\log z$ as a multi-valued function.

   Solution:

   $\log(\frac{1}{z}) = \ln|\frac{1}{z}| + iarg(\frac{1}{z}) = \ln\frac{1}{|z|} + iarg(z^{-1}) = -\ln|z| - iarg(z) = -\log(z)$

5. Compute $4^{\frac{1}{2}}$ using the definition of complex power function

   Solution:

$$
\begin{aligned}
4^{\frac{1}{2}} &= e^{\frac{1}{2}\log 4} \\
&= e^{\frac{1}{2}(\ln 4 + iarg(4))} \\
&= e^{\frac{1}{2}(2\ln 2 + 2\pi ki)} \\
&= e^{\ln 2 + \pi ki} \\
&= 2e^{\pi ki} \\
&= \pm 2
\end{aligned}
$$

6. Prove that $f(z) = z^c, (z \neq 0)$ is a single-valued function if and only if $c \in \mathbb{Z}$.

   Solution: Let $c = a + bi$

$$
\begin{aligned}
z^c &= e^{c\log z} \\
&= e^{(a+bi)(\ln|z| + iargz)} \\
&= e^{(a+bi)(\ln|z| + i(Argz + 2\pi k)} \\
&= e^{a\ln|a| - b(Argz+2\pi k)} e^{(b\ln|z| + aArg(z) + 2\pi ak)i}
\end{aligned}
$$

   The function is single-valued $\iff$ its value doesn't depend on $k \iff b = 0$ and ak is always an integer $\iff b = 0$ and $a \in \mathbb{Z} \iff c \in \mathbb{Z}$.

7. (0 Credit, but you are encouraged to do it if you have some knowledge of group theory.)

   Prove the set of all complex numbers of norm 1 form a group under multiplication, and this group is isomorphic to $SO_2(\mathbb{R}) = O_2(\mathbb{R}) \cap SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) | A^T = A^{-1}, \det(A) = 1\}$.

   Denote $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

   Indeed we can prove S^1 is a subgroup of $\mathbb{C}^\times$:

   1. For any $z_1, z_2 \in S^1$, $|Z_1 z_2| = |z_1||z_2| = 1$, so $z_1 z_2 \in S^1$.
   2. The identity element $1 \in S^1$
   3. If $z \in S^1$, then $|\frac{1}{z}| = \frac{1}{|z|} = 1$, so $\frac{1}{z} \in S^1$.

We conclude S^1 is a subgroup of $\mathbb{C}$.

Next, we can define a surjective homomorphism $f_1 : \mathbb{R} \longrightarrow S^1$ by $f_1(r) = e^{ir}$, and $\ker(f_1) = 2\pi\mathbb{Z}$. By First isomorphism Theorem $S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$.

On the other hand, Assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO_2$, then $det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc = 1$, so

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

So $a = d$ and $b = -c$, the matrix becomes $\begin{bmatrix} a & -c \\ c & a \end{bmatrix}$, with $a^2 + c^2 = 1$. We know for a pair of real numbers a, c satisfying $a^2 + c^2 = 1$, the angle x whose terminal edge passing through (a, c) has $\cos x = a$ and $\sin x = c$, hence the matrix can be written as $\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

The following surjective homomorphism

$$f_2 : \mathbb{R} \longrightarrow SO_2$$

$$x \mapsto \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

also has kernel $\ker(f_2) = 2\pi\mathbb{Z}$, so $SO_2 \cong \mathbb{R}/(2\pi\mathbb{Z}) \cong S^1$