1. Show that \( f'(z) \) doesn’t exist at any point if \( f(z) = z + \bar{z} \).

**Solution:** \( f(z) = z + \bar{z} = x + yi + x - yi = 2x, \) so \( u(x, y) = 2x, v(x, y) = 0 \). Then \( u_x = 2 \neq 0 = v_y \) at any point, the Cauchy-Riemann Equations are never satisfied, so \( f' \) doesn’t exist anywhere.

2. Determine when \( f'(z) \) exists if \( f(z) = z \text{Re}(z) \), where \( \text{Re}(z) \) denotes the real part of \( z \).

**Solution:** \( f'(z) = z \text{Re}(z) = (x+yi)x = x^2 + xyi, \) so \( u(x, y) = x^2, v(x, y) = xy \).

\( u_x = 2x, u_y = 0, v_x = y, v_y = x \) exists and are continuous, and the Cauchy-Riemann Equations hold if and only if \( (x, y) = (0, 0) \), so \( f' \) exists only at \( z = 0 \).

3. Prove that \( f(z) = (3x + y) + i(3y - x) \) is entire.

**Solution:** \( u_x = 3 = v_y \) and \( u_y = 1 = -(v_x) \), so the partial derivatives are continuous and satisfy Cauchy-Riemann equations everywhere, we conclude \( f'(z) \) exists everywhere, so \( f \) is entire.

4. (i). \( z = x + yi \in \mathbb{C}, \) prove \( x = \frac{z + \bar{z}}{2} \) and \( y = \frac{z - \bar{z}}{2i} \)

**Solution:**
\[
\frac{z + \bar{z}}{2} = \frac{x + yi + x - yi}{2} = \frac{2x}{2} = x, \quad \text{and} \quad \frac{z - \bar{z}}{2i} = \frac{x + yi - x - yi}{2i} = \frac{2yi}{2i} = y
\]

(ii). We define \( \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \). Prove if \( f'(z) \) exists, then \( \frac{\partial f}{\partial z} = 0. \)

**Solution:** If \( f'(z) \) exists, then \( u_x = v_y \) and \( u_y = -v_x. \)

\[
\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f
= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)
= \frac{1}{2}\left(u_x + v_xi + i(u_y + v_yi)\right)
= \frac{1}{2}\left(u_x - v_y + i(u_y + v_x)\right)
= 0
\]
5. Derive the polar form of Cauchy-Riemann Equations. (Recall that $x = r \cos \theta$ and $y = r \sin \theta$).

Solution:

By the chain rule, we have

\[
\begin{align*}
    u_r &= u_x \cos \theta + u_y \sin \theta \\
    u_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\
    v_r &= v_x \cos \theta + v_y \sin \theta \\
    v_\theta &= -v_x r \sin \theta + v_y r \cos \theta
\end{align*}
\]

We see $ru_r = rv_x \cos \theta + v_y \sin \theta = v_\theta$ and $rv_r = v_x r \cos \theta + v_y r \sin \theta = -u_\theta$. We conclude

\[
\begin{align*}
    ru_r &= v_\theta \\
    rv_r &= -u_\theta
\end{align*}
\]

6. Derive the polar form of Laplace’s equation $u_{xx} + u_{yy} = 0$.

Solution:

\[
u_r = u_x x_r + u_y y_r = u_x \cos \theta + u_y \sin \theta.
\]

\[
u_{rr} = (u_x)_r \cos \theta + (u_y)_r \sin \theta \\
= (u_{xx} x_r + u_{yx} y_r) \cos \theta + (u_{xy} x_r + u_{yy} y_r) \\
= u_{xx} \cos^2 \theta + 2u_{xy} \cos \theta \sin \theta + u_{yy} \sin^2 \theta
\]

\[
u_\theta = u_x x_\theta + u_y y_\theta = -u_x r \sin \theta + u_y r \cos \theta
\]

\[
u_{\theta\theta} = -u_x r \cos \theta - (u_x)_\theta r \sin \theta - u_y r \sin \theta + (u_y)_\theta r \cos \theta \\
= -u_x r \cos \theta - (u_{xx} x_\theta + u_{yx} y_\theta) r \sin \theta - u_y r \sin \theta + (u_{xy} x_\theta + u_{yy} y_\theta) r \cos \theta \\
= -r(u_x \cos \theta + u_y \sin \theta) + r^2(u_{xx} \sin^2 \theta - 2u_{xy} \cos \theta \sin \theta) + u_{yy} \cos^2 \theta
\]
Note that
\[
    r^2 u_{rr} + ru_r + u_{\theta\theta} = r^2(2u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta) + r(u_x \cos\theta + u_y \sin\theta) \\
    - r(u_x \cos\theta + u_y \sin\theta) + r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta) + u_{yy}\cos^2\theta \\
    = u_{xx} + u_{yy}
\]

So the Laplace Equation in polar form is
\[
r^2 u_{rr} + ru_r + u_{\theta\theta} = 0
\]

7. If \( u(x, y) \) is a harmonic function on a domain \( D \), we say \( v(x, y) \) is a harmonic conjugate of \( u(x, y) \) if the complex function \( f(z) = u(x, y) + v(x, y)i \) has derivative. Prove that if \( v(x, y) \) and \( w(x, y) \) are both harmonic conjugates of \( u(x, y) \) on \( D \), then \( v(x, y) \) and \( w(x, y) \) are differed by a constant.

**Solution:** If \( v(x, y) \) and \( w(x, y) \) are both harmonic conjugates of \( u(x, y) \) on \( D \), then \( f(z) = u(x, y) + v(x, y)i \) and \( g(z) = u(x, y) + w(x, y)i \) are analytic on \( D \), so their difference \( F(z) = f(z) - g(z) = 0 + (v(x, y) - w(x, y))i \) is also analytic on \( D \). So we know the derivative \( F'(z) = 0_x + (v - w)_x i = 0_x - 0_y i = 0 \) on \( D \), which implies \( F \) is constant, so \( v - w \) is a constant.