1. Show that $f'(z)$ doesn't exist at any point if $f(z) = z + \bar{z}$.

   **Solution**: $f(z) = z + \bar{z} = x + yi + x - yi = 2x$, so $u(x, y) = 2x$, $v(x, y) = 0$. Then $u_x = 2 \neq 0 = v_y$ at any point, the Cauchy-Riemann Equations are never satisfied, so $f'$ doesn't exist anywhere.

2. Determine when $f'(z)$ exists if $f(z) = zRe(z)$, where $Re(z)$ denotes the real part of $z$.

   **Solution**: $f'(z) = zRe(z) = (x+yi)x = x^2 + xyi$, so $u(x, y) = x^2$, $v(x, y) = xy$.

   $u_x = 2x, u_y = 0, v_x = y, v_y = x$ exists and are continuous, and the Cauchy-Riemann Equations hold if and only if $(x, y) = (0, 0)$, so $f'$ exists only at $z = 0$.

3. Prove that $f(z) = (3x + y) + i(3y - x)$ is entire.

   **Solution**: $u_x = 3 = v_y$ and $u_y = 1 = -(v_x)$, so the partial derivatives are continuous and satisfy Cauchy-Riemann equations everywhere, we conclude $f'(z)$ exists everywhere, so $f$ is entire.

4. (i). $z = x + yi \in \mathbb{C}$, prove $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

   **Solution**:

   $\frac{z+\bar{z}}{2} = \frac{x+yi+x-yi}{2} = \frac{2x}{2} = x$, and $\frac{z-\bar{z}}{2i} = \frac{x+yi-x+yi}{2i} = \frac{2yi}{2i} = y$

   (ii). We define $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Prove if $f'(z)$ exists, then $\frac{\partial f}{\partial \bar{z}} = 0$.

   **Solution**: If $f'(z)$ exists, then $u_x = v_y$ and $u_y = -v_x$.

$$
\begin{aligned}
\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f \\
&= \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) \\
&= \frac{1}{2}(u_x + v_x i + i(u_y + v_y i)) \\
&= \frac{1}{2}(u_x - v_y + i(u_y + v_x)) \\
&= 0
\end{aligned}
$$

5. Derive the polar form of Cauchy-Riemann Equations. (Recall that $x = r\cos\theta$ and $y = r\sin\theta$).

   **Solution**:

   By the chain rule, we have

$$\begin{cases} u_r = u_x \cos\theta + u_y \sin\theta \\ u_\theta = -u_x r\sin\theta + u_y r\cos\theta \\ v_r = v_x \cos\theta + v_y \sin\theta \\ v_\theta = -v_x r\sin\theta + v_y r\cos\theta \end{cases}$$

   We see $ru_r = ru_x\cos\theta + u_y\sin\theta = rv_y\cos\theta - v_x r\sin\theta = v_\theta$ and $rv_r = v_x r\cos\theta + v_y r\sin\theta = -u_y r\cos\theta + u_x r\sin\theta = -u_\theta$. We conclude

$$\begin{cases} ru_r = v\theta \\ rv_r = -u_\theta \end{cases}$$

6. Derive the polar form of Laplace's equation $u_{xx} + u_{yy} = 0$.

   **Solution**:

   $u_r = u_x x_r + u_y y_r = u_x\cos\theta + u_y\sin\theta$.

$$\begin{aligned} u_{rr} &= (u_x)_r\cos\theta + (u_y)_r\sin\theta \\ &= (u_{xx}x_r + u_{yx}y_r)\cos\theta + (u_{xy}x_r + u_{yy}y_r) \\ &= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta \end{aligned}$$

   $u_\theta = u_x x_\theta + u_y y_\theta = -u_x r\sin\theta + u_y r\cos\theta$

$$\begin{aligned} u_{\theta\theta} &= -u_x r\cos\theta - (u_x)_\theta r\sin\theta - u_y r\sin\theta + (u_y)_\theta r\cos\theta \\ &= -u_x r\cos\theta - (u_{xx}x_\theta + u_{yx}y_\theta)r\sin\theta - u_y r\sin\theta + (u_{xy}x_\theta + u_{yy}y_\theta)r\cos\theta \\ &= -r(u_x\cos\theta + u_y\sin\theta) + r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta) + u_{yy}\cos^2\theta \end{aligned}$$

Note that

$$\begin{aligned}
&r^2u_{rr} + ru_r + u_{\theta\theta} \\
=&r^2(u_x x\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta) + r(u_x\cos\theta + u_y\sin\theta) \\
&- r(u_x\cos\theta + u_y\sin\theta) + r^2(u_{xx}\sin^2\theta - 2u_{xy}\cos\theta\sin\theta) + u_{yy}\cos^2\theta \\
=&u_{xx} + u_{yy}
\end{aligned}$$

So the Laplace Equation in polar form is

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0$$

7. If $u(x, y)$ is a harmonic function on a domain $D$, we say $v(x, y)$ is a harmonic conjugate of $u(x, y)$ if the complex function $f(z) = u(x, y) + v(x, y)i$ has derivative. Prove that if $v(x, y)$ and $w(x, y)$ are both harmonic conjugates of $u(x, y)$ on $D$, then $v(x, y)$ and $w(x, y)$ are differed by a constant.

   **Solution**: If $v(x, y)$ and $w(x, y)$ are both harmonic conjugates of $u(x, y)$ on $D$, then $f(z) = u(x, y) + v(x, y)i$ and $g(z) = u(x, y) + w(x, y)i$ are analytic on $D$, so their difference $F(z) = f(z) - g(z) = 0 + (v(x, y) - w(x, y))i$ is also analytic on $D$. So we know the derivative $F'(z) = 0_x + (v - w)_x i = 0_x - 0_y i = 0$ on $D$, which implies $F$ is constant, so $v - w$ is a constant.