1. $f(z) = \frac{1-e^{2z}}{z^4}$. What is the order of the pole $z = 0$?

   Solution:

   $$f(z) = \frac{1-e^{2z}}{z^4} = \frac{1}{z^4}(1 - \sum_{n=0}^{+\infty} \frac{1}{n!}(2z)^n) = -\sum_{n=1}^{+\infty} \frac{2^n}{n!} z^{n-4} = \sum_{n=-3}^{+\infty} \frac{2^{n+4}}{(n+4)!} z^n$$

   So the order of the pole is -3.

2. $f(z)$ is analytic at z_0. $g(z) = \frac{f(z)}{z-z_0}$. What type of singularity is z_0 for the function g?

   Solution:

   If $f(z_0) \neq 0$, it is a pole.

   If $f(z_0) = 0$, the Taylor expansion of f at z_0 is $f(z) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$, so the Laurent expansion of $g(z)$ around z_0 is

   $$g(z) = \frac{0}{z - z_0} f(z) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-1} = \sum_{n=0}^{+\infty} \frac{f^{(n+1)}(z_0)}{(n+1)!}(z - z_0)^n$$

   So z_0 is a removable singular point.

3. Compute the residue of $\frac{1}{z^2(z+1)^2}$ at $z = 0$.

   Solution:

   Let $\phi(z) = \frac{1}{(z+1)^2}$, then $\frac{1}{z^2(z+1)^2} = \frac{\phi(z)}{z^2}$ with $\phi(z)$ analytic at $z = 0$ and $\phi(0) \neq 0$, so the residue of $\frac{1}{z^2(z+1)^2}$ at $z = 0$ is

   $$\frac{\phi'(0)}{(2-1)!} = \phi'(0) = -2$$

4. Prove $z = 0$ is a pole for $f(z) = \frac{1}{z(e^z-1)}$ and compute the residue at $z = 0$.

   Solution:

   $\frac{e^z-1}{z} = \frac{1}{z}\sum_{n=1}^{+\infty} \frac{1}{n!} z^n = \sum_{n=1}^{+\infty} \frac{1}{n!} z^{n-1} = 1 + \sum_{n=1}^{+\infty} \frac{1}{(n+1)!} z^n$, so $g(z) = \frac{e^z-1}{z}$ is analytic at 0 and $g(0) = 1 \neq 0$, $g'(0) = \frac{1}{2}$

$$f(z) = \frac{1}{z^2 g(z)} = \frac{\frac{1}{g(z)}}{z^2}$$

where $\frac{1}{g(z)}$ is analytic at 0 and $\frac{1}{g(0)} = \frac{1}{1} = 1 \neq 0$, so 0 is a pole of order 2 for $f(z)$, and

$$Res_{z=0} f = (\frac{1}{g(z)})'|_{z=0} = -\frac{g'(0)}{g(0)^2} = -\frac{1}{2}$$

5. Show that $z = 0$ is a simple pole for $f(z) = \frac{1}{\sin z}$, and compute the residue at $z = 0$.

   Solution:

   $\sin 0 = 0$, but $(\sin)'(0) = \cos 0 = 1 \neq 0$, so $z = 0$ is a zero of order 1 for $\sin z$. Note the constant function 1 is analytic and nonzero at $z = 0$, we conclude $\frac{1}{\sin z}$ has a simple pole at $z = 0$. The residue is

$$\frac{1}{(sin)'(0)} = \frac{1}{\cos 0} = 1$$

6. p and q are functions that are analytic at z_0, and $p(z_0) \neq 0$, $q(z_0) = 0$. Show that if z_0 is a pole of order m for $f(z) = \frac{p(z)}{q(z)}$, then z_0 is a zero of order m for q.

   Solution:

   z_0 is a pole of order m for $f(z) = \frac{p(z)}{q(z)}$, there exists ϕ analytic at z_0, $\phi(z_0) \neq 0$ such that

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m}$$

   This means

$$q(z) = (z - z_0)^m \frac{p(z)}{\phi(z)}$$

   and $\frac{p(z)}{\phi(z)}$ is analytic at z_0 and nonzero at z_0, we conclude z_0 is a zero of order m for q

7. C is the positively oriented circle $|z| = e$. Evaluate

$$\int_C \tan z \, dz$$

   Solution:

$\tan z = \frac{\sin z}{\cos z}$, z_0 is a singular point of $\tan z$ if and only if z_0 is a zero for $\cos z$. For any z_0 a zero of order 1 of $\cos z$, $\cos z_0 = 0$, $\sin^2 z_0 = 1 - \cos^2 z_0 = 1$, so $\sin z_0 \neq 0$. We see z_0 is a simple pole for $\tan z$.

$$Res_{z=z_0} \tan z = \frac{\sin z_0}{(\cos)'(z_0)} = \frac{\sin z_0}{-\sin z_0} = -1$$

Inside C, the singular points are $\pm\frac{\pi}{2}$, so

$$\int_C \tan z \, dz = 2\pi i(Res_{z=\frac{\pi}{2}}) \tan z + Res_{z=\frac{-\pi}{2}}) \tan z = 2\pi i(-1-1) = -4\pi i$$

8. $q(z)$ is a function analytic at z_0, $q(z_0) = 0$, $q'(z_0) \neq 0$. Show that z_0 is a pole of order 2 of $f(z) = \frac{1}{q(z)^2}$, and prove the residue of f at z_0 is $-\frac{q''(z_0)}{(q'(z_0))^3}$

Solution:

z_0 is a zero of order 1 for q, so $q(z) = (z - z_0)g(z)$ for some $g(z)$ analytic at z_0 and $g(z_0) \neq 0$.

$$f(z) = \frac{1}{(q(z))^2} = \frac{\frac{1}{g(z)^2}}{(z - z_0)^2}$$

and $\frac{1}{g(z)^2}$ is analytic and nonzero at z_0, so we see z_0 is a pole of order 2 for f.

Next we are going to compute $Res_{z=z_0} f$

$q'(z) = (z - z_0)g'(z) + g(z)$ and $q''(z) = 2g'(z) + (z - z_0)g''(z)$, so

$$q'(z_0) = g(z), q''(z_0) = 2g'(z_0)$$

$$Res_{z=z_0} f = (\frac{1}{g(z)^2})'|_{z=z_0} = -\frac{2g'(z_0)}{(g(z_0))^3} = -\frac{q''(z_0)}{(q'(z_0))^3}$$