1. \( f(z) = \frac{1 - e^{2z}}{z^4} \). What is the order of the pole \( z = 0 \)?

**Solution:**

\[
f(z) = \frac{1 - e^{2z}}{z^4} = \frac{1}{z^4} \left(1 - \sum_{n=0}^{+\infty} \frac{1}{n!} (2z)^n \right) = \frac{1}{z^4} \sum_{n=1}^{+\infty} \frac{2^n}{n!} z^{n-4} = \sum_{n=-3}^{+\infty} \frac{2^{n+4}}{(n+4)!} z^n
\]

So the order of the pole is \(-3\).

2. \( f(z) \) is analytic at \( z_0 \). \( g(z) = \frac{f(z)}{z-z_0} \). What type of singularity is \( z_0 \) for the function \( g \)?

**Solution:**

If \( f(z_0) \neq 0 \), it is a pole.

If \( f(z_0) = 0 \), the Taylor expansion of \( f \) at \( z_0 \) is

\[
f(z) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n
\]

so the Laurent expansion of \( g(z) \) around \( z_0 \) is

\[
g(z) = \frac{0}{z-z_0} f(z) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^{n-1} = \sum_{n=0}^{+\infty} \frac{f^{(n+1)}(z_0)}{(n+1)!} (z-z_0)^n
\]

So \( z_0 \) is a removable singular point.

3. Compute the residue of \( \frac{1}{z^2(z+1)^2} \) at \( z = 0 \).

**Solution:**

Let \( \phi(z) = \frac{1}{(z+1)^2} \), then \( \frac{1}{z^2(z+1)^2} = \frac{\phi(z)}{z^2} \) with \( \phi(z) \) analytic at \( z = 0 \) and \( \phi(0) \neq 0 \), so the residue of \( \frac{1}{z^2(z+1)^2} \) at \( z = 0 \) is

\[
\frac{\phi'(0)}{(2-1)!} = \phi'(0) = -2
\]

4. Prove \( z = 0 \) is a pole for \( f(z) = \frac{1}{z(e^z-1)} \) and compute the residue at \( z = 0 \).

**Solution:**

\[
\frac{e^{z-1}}{z} = \frac{1}{z} \sum_{n=1}^{+\infty} \frac{1}{n!} z^n = \sum_{n=1}^{+\infty} \frac{1}{n!} z^{n-1} = 1 + \sum_{n=1}^{+\infty} \frac{1}{(n+1)!} z^n
\]

so \( g(z) = \frac{e^{z-1}}{z} \) is analytic at 0 and \( g(0) = 1 \neq 0, g'(0) = \frac{1}{2} \).
\[ f(z) = \frac{1}{z^2 g(z)} = \frac{1}{g(z)} \]

where \( \frac{1}{g(z)} \) is analytic at 0 and \( \frac{1}{g(0)} = \frac{1}{1} = 1 \neq 0 \), so 0 is a pole of order 2 for \( f(z) \), and

\[ \text{Res}_{z=0} f = \left( \frac{1}{g(z)} \right)'|_{z=0} = -\frac{g'(0)}{g(0)^2} = -\frac{1}{2} \]

5. Show that \( z = 0 \) is a simple pole for \( f(z) = \frac{1}{\sin z} \), and compute the residue at \( z = 0 \).

**Solution:**

\( \sin 0 = 0 \), but \( (\sin)'(0) = \cos 0 = 1 \neq 0 \), so \( z = 0 \) is a zero of order 1 for \( \sin z \). Note the constant function 1 is analytic and nonzero at \( z = 0 \), we conclude \( \frac{1}{\sin z} \) has a simple pole at \( z = 0 \). The residue is

\[ \frac{1}{(\sin)'(0)} = \frac{1}{\cos 0} = 1 \]

6. \( p \) and \( q \) are functions that are analytic at \( z_0 \), and \( p(z_0) \neq 0, q(z_0) = 0 \). Show that if \( z_0 \) is a pole of order \( m \) for \( f(z) = \frac{p(z)}{q(z)} \), then \( z_0 \) is a zero of order \( m \) for \( q \).

**Solution:**

\( z_0 \) is a pole of order \( m \) for \( f(z) = \frac{p(z)}{q(z)} \), there exists \( \phi \) analytic at \( z_0, \phi(z_0) \neq 0 \) such that

\[ \frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \]

This means

\[ q(z) = (z - z_0)^m \frac{p(z)}{\phi(z)} \]

and \( \frac{p(z)}{\phi(z)} \) is analytic at \( z_0 \) and nonzero at \( z_0 \), we conclude \( z_0 \) is a zero of order \( m \) for \( q \)

7. \( C \) is the positively oriented circle \( |z| = e \). Evaluate

\[ \int_C \tan z \, dz \]

**Solution:**
\[
\tan z = \frac{\sin z}{\cos z}, \quad z_0 \text{ is a singular point of } \tan z \text{ if and only if } z_0 \text{ is a zero for } \cos z.
\]
For any \( z_0 \) a zero of order 1 of \( \cos z \), \( \cos z_0 = 0 \), \( \sin^2 z_0 = 1 - \cos^2 z_0 = 1 \), so \( \sin z_0 \neq 0 \). We see \( z_0 \) is a simple pole for \( \tan z \).

\[
\text{Res}_{z=z_0} \tan z = \frac{\sin z_0}{(\cos')'(z_0)} = \frac{\sin z_0}{-\sin z_0} = -1
\]

Inside \( C \), the singular points are \( \pm \frac{\pi}{2} \), so
\[
\int_C \tan z \, dz = 2\pi i (\text{Res}_{z=\frac{\pi}{2}} \tan z + \text{Res}_{z=-\frac{\pi}{2}} \tan z) = 2\pi i (-1 - 1) = -4\pi i
\]

8. \( q(z) \) is a function analytic at \( z_0 \), \( q(z_0) = 0 \), \( q'(z_0) \neq 0 \). Show that \( z_0 \) is a pole of order 2 of \( f(z) = \frac{1}{q(z)^2} \), and prove the residue of \( f \) at \( z_0 \) is \( -\frac{q''(z_0)}{(q'(z_0))^3} \).

\textbf{Solution:}

\( z_0 \) is a zero of order 1 for \( q \), so \( q(z) = (z - z_0)g(z) \) for some \( g(z) \) analytic at \( z_0 \) and \( g(z_0) \neq 0 \).

\[
f(z) = \frac{1}{(q(z))^2} = \frac{1}{g(z)^2 (z - z_0)^2}
\]

and \( \frac{1}{g(z)^2} \) is analytic and nonzero at \( z_0 \), so we see \( z_0 \) is a pole of order 2 for \( f \).

Next we are going to compute \( \text{Res}_{z=z_0} f \)

\( q'(z) = (z - z_0)g'(z) + g(z) \) and \( q''(z) = 2g'(z) + (z - z_0)g''(z) \), so

\[
q'(z_0) = g(z_0), q''(z_0) = 2g'(z_0)
\]

\[
\text{Res}_{z=z_0} f = (\frac{1}{g(z)^2})|_{z=z_0} = \frac{2g'(z_0)}{(g(z_0))^3} = -\frac{q''(z_0)}{(q'(z_0))^3}
\]