1. C is the positively oriented circle $|z| = 3$. Evaluate the following integrals:

   (i). $\int_C \frac{e^{-z}}{z^2}\, dz$

   Solution: Let $f(z) = \frac{e^{-z}}{z^2}$. $f(z)$ is analytic on and inside C except at $z_0 = 0$. The Laurent series expansion of f on $0 < |z| < \infty$ is

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{n!}(-z)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{n-2}$$

   So $Res_{z=0}f = \frac{(-1)^1}{1!} = -1$, and

$$\int_C \frac{e^{-z}}{z^2}\, dz = 2\pi i(-1) = -2\pi i$$

   (ii). $\int_C z^2 e^{\frac{1}{z}}\, dz$

   Solution: Let $f(z) = z^2 e^{\frac{1}{z}}$. f is analytic on and inside C except at $z_0 = 0$. The Laurent series expansion of f on $0 < |z| < \infty$ is

$$z^2 e^{\frac{1}{z}} = z^2\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{1}{z})^n = \sum_{n=0}^{\infty}\frac{1}{n!}z^{2-n}$$

   So $Res_{z=0}f = \frac{1}{3!} = \frac{1}{6}$

$$\int_C z^2 e^{\frac{1}{z}}\, dz = 2\pi i \times \frac{1}{6} = \frac{1}{3}\pi i$$

2. C is the positively oriented circle $|z| = 3$. Evaluate

$$\int_C \frac{1}{1+z^2}\, dz$$

   by:

   (i). Using the singular points inside C.

   Solution: Let $f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$. f is analytic on and inside C except at $z_0 = \pm i$.

$Res_{z=i}f = \frac{1}{i+i} = \frac{1}{2i}$, $Res_{z=-i}f = \frac{1}{-i-i} = -\frac{1}{2i}$.

$$\int_C \frac{1}{1+z^2}\, dz = 2\pi i(Res_{z=i}f + Res_{z=-1}f) = 0$$

(ii). Using the residue at infinity.

Solution: $\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2}\frac{1}{1+(\frac{1}{z})^2} = \frac{1}{1+z^2}$, which is analytic at 0, so

$$Res_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = 0$$

$$\int_C \frac{1}{1+z^2}\, dz = 2\pi i \times 0 = 0$$

3. $p(z) = a_0 + a_1 z + ... + a_n z^n$, $a_n \neq 0$. $q(z) = b_0 + b_1 z + ... + b_m z^m$, $b_m \neq 0$. If $m \geq n+2$ and all the roots of $q(z) = 0$ are inside a simple closed contour C, prove

$$\int_C \frac{p(z)}{q(z)}\, dz = 0$$

Solution:

$$\begin{aligned}\frac{1}{z^2}f(\frac{1}{z}) &= \frac{1}{z^2}\frac{a_0 + ... + \frac{a_n}{z^n}}{b_0 + ... + \frac{b_m}{z^m}} \\ &= \frac{1}{z^2}\frac{a_0 z^m + ... + a_n z^{m-n}}{b_0 z^m + ... + b_m} \\ &= \frac{a_0 z^{m-2} + ... + a_n z^{m-n-2}}{b_0 z^m + ... + b_m}\end{aligned}$$

So $\frac{1}{z^2}f(\frac{1}{z})$ is analytic at 0, $Res_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = 0$, so

$$\int_C \frac{p(z)}{q(z)}\, dz = 2\pi i \times 0 = 0$$

4. C is a positively oriented simple closed curve. f is a function which is analytic on and outside of C except at finitely many points $z_1, ..., z_n$. Prove

$$\int_C f(z)\, dz = 2\pi i[Res_{z=0}(\frac{1}{z^2}f(\frac{1}{z})) - \sum_{k=1}^{n} Res_{z=z_k} f(z)]$$

Solution: Construct positively oriented circles L_i: $|z - z_k| < r_i$ such that the disks $|z - z_k| \leq r_i$ are disjoint to each other and all of them are disjoint from C. Let C' be a positively oriented contour enclosing C and all L_k in the interior. We see:

$$\begin{aligned} 2\pi i(Res_{z=0}(\frac{1}{z^2}f(\frac{1}{z})) &= \int_{C'} f(z)\,dz \\ &= \int_C f(z)\,dz + \sum_{k=1}^{n}\int_{L_k} f(z)\,dz \\ &= \int_C f(z)\,dz + 2\pi i\sum_{k=1}^{n} Res_{z=z_k} \end{aligned}$$

So

$$\int_C f(z)\,dz = 2\pi i[Res_{z=0}(\frac{1}{z^2}f(\frac{1}{z})) - \sum_{k=1}^{n} Res_{z=z_k} f(z)]$$

5. Prove $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z_0 = 0$, and then extend f to an entire function.

Solution:

The Laurent Expansion of $\frac{\sin z}{z}$ at $z_0 = 0$ is

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}, (0 < |z| < \infty)$$

which has no negative powers, so $z_0 = 0$ is a removable singularity. we can make f analytic at $z_0 = 0$ by defining $f(0) = \frac{(-1)^0}{1!} = 1$, and f is analytic everywhere else, so we get an entire function by this extension.