COMPLEX NUMBERS

The concept of complex numbers came from the wish of solving the equation

$$x^2+1=0.$$

We know this equation has no real number solution since the square of real numbers are all nonnegative.

The way to solve for this equation is to imagine it has a root, and we call it i. (i stands for imaginary)
If i solves $x^2+1=0$, then i is a "number" such that
$i^2+1=0$.

Since we add the number i to $\mathbb{R}$. we then need to care about how to make the sum of i and real numbers, product of i and real numbers meaningful.

We define a complex number to be a number of the form $z=x+yi$, where x and y are real numbers.

Given two complex numbers $z_1=x_1+y_1 i$ and $z_2=x_2+y_2 i$, we define their sum to be

$$z_1+z_2=(x_1+x_2)+(y_1+y_2)i$$

and their product to be

$$z_1\cdot z_2=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)i$$

The set of all the complex numbers with the above two operations form a field, called the Field of Complex Numbers, and denoted by $\mathbb{C}$.

We can define subtraction & division of complex numbers based on the definition of addition and multiplication:

$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$

If $z_2 \neq 0$, $\dfrac{z_1}{z_2} = \dfrac{x_1 + y_1 i}{x_2 + y_2 i} = \dfrac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)}$

$$= \frac{(x_1x_2 + y_1y_2) + (x_2y_1 - x_1y_2)i}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} i$$

In particular, given a nonzero complex number $z = x + yi$, its multiplicative inverse is $z^{-1} = \dfrac{1}{z} = \dfrac{1}{x+yi} = \dfrac{x - yi}{x^2+y^2}$

So this indicates we can understand $\dfrac{z_1}{z_2}$ as $z_1 \cdot z_2^{-1}$.

Example. $(3+2i) + (4-3i) = (3+4) + (2-3)i = 7 - i$

$(3+2i)(4-3i) = (3\times4 + 2\times3) + (2\times4 - 3\times3)i$
$= 18 - i$

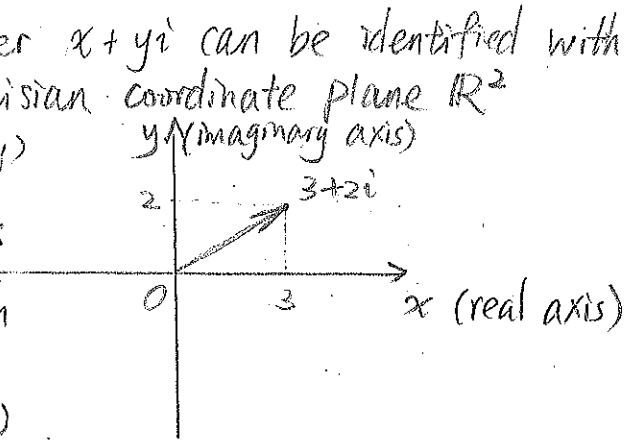
$$\frac{3+2i}{4-3i} = \frac{(3+2i)(4+3i)}{(4-3i)(4+3i)} = \frac{6+17i}{4^2+3^2} = \frac{6}{25} + \frac{17}{25}i$$

$$(4-3i)^{-1} = \frac{1}{4-3i} = \frac{4+3i}{4^2+3^2} = \frac{4}{25} + \frac{3}{25}i$$

Geometric Presentation of Complex Numbers:

Each complex number $x + yi$ can be identified with the point (x, y) on the Cartisian coordinate plane $\mathbb{R}^2$

Also recall that (x, y) is identified with its "position vector", which starts at $(0,0)$ and terminates at (x, y).



Given a complex number $z = x + yi$, we call x the real part of z and y the imaginary part of z, denote by $x = \mathrm{Re}(z)$; $y = \mathrm{Im}(z)$.

We see z is identified with the vector whose first entry is $\mathrm{Re}(z)$ and second entry is $\mathrm{Im}(z)$.

Recall that given a vector $\vec{v} = (x, y)$, its length is defined as $|\vec{v}| = \sqrt{x^2 + y^2}$.

Using this idea, we find a way to measure the size of a complex number, by the length of its corresponding vector.

Definition. The modulus (or absolute value) of a complex number $z = x + yi$ is $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \sqrt{x^2 + y^2}$

Geometrically, $|z|$ stands for the distance between (x, y) and $(0, 0)$ on the plane, which restricts to the real numbers gives the absolute value of a real number.

Proposition. $|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$, $\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|$, $\mathrm{Im}(z) \leq |\mathrm{Im}(z)| \leq |z|$

Example. Which of $z_1 = -3 + 2i$ and $z_2 = 1 + 4i$ is closer to the origin?

$|z_1|^2 = (-3)^2 + 2^2 = 13$, $|z_2|^2 = 1^2 + 4^2 = 17$

so $|z_1| = \sqrt{13} < \sqrt{17} = |z_2|$, z_1 is closer to origin

Proposition (Triangle Inequality). For any complex numbers z_1 and z_2.

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$, and equality holds if and only if z_1, z_2 are on a same ray starting from O.

(ii) $|z_1 - z_2| \geq ||z_1| - |z_2||$, and equality holds if and only if z_1, z_2 are on a same ray starting from O.

Proof. These follows directly from the triangle inequality of vectors.

Remark. We can extend the triangle inequality to n complex numbers $z_1, z_2, \ldots, z_n$:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

Proposition. For any $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1| \cdot |z_2|$

Proof. let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ then $|z_1|^2 = x_1^2 + y_1^2$, $|z_2|^2 = x_2^2 + y_2^2$

$$\begin{aligned}|z_1 z_2|^2 = |(x_1 + y_1 i)(x_2 + y_2 i)|^2 &= |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i|^2 \\ &= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 \\ &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) = |z_1|^2 \cdot |z_2|^2\end{aligned}$$

Definition. $z = x + yi$ is a complex number, we define its complex conjugate to be $\bar{z} = x - yi$

Properties: (i) If $z \in \mathbb{R}$, then $z = \bar{z}$

(ii) $z \cdot \bar{z} = |z|^2$

(iii) $|z| = |\bar{z}|$

(iv) On the complex plane, z and $\bar{z}$ are symmetric along the real axis

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

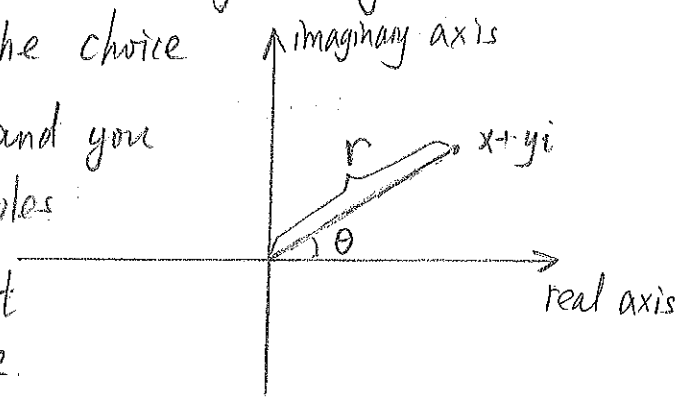
$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(vi) $\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2}$

(vii) $\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$, $\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$

EXPONENTIAL FORM OF COMPLEX NUMBERS

Recall that we can identify the complex number $z = x + yi$ with the point (x, y) on the plane. Given a point (x, y) in the Cartisian plane, we can describe it in the polar coordinates by (r, θ), where r is the distance between (x, y) and the origin. $r = \sqrt{x^2 + y^2}$, and θ is the angle whose terminal edge passes through (x, y), so $x = r\cos\theta$, $y = r\sin\theta$. Note that the choice of θ is not unique, and you can add integral multiples of 2π to θ to represent the same terminal edge.



This observation motivates us to represent a complex number by polar coordinates: If $z = x + yi$, we write

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

where (r, θ) is the polar coordinate corresponding to the Cartisian coordinate (x, y)

Definition We call θ an argument of z, and $\arg z$ is the set of all such θ. We define the principal value of $\arg z$ to be the value in $\arg z \cap (-\pi, \pi]$, and denote by $\operatorname{Arg} z$.

Remark. $\arg z = \{\operatorname{Arg} z + 2\pi k \in \mathbb{R} \mid k \in \mathbb{Z}\}$

Example. If $z = 1+i$, $\mathrm{Arg}\, z = \frac{\pi}{4}$.

If $z = 1-i$, $\mathrm{Arg}\, z = -\frac{\pi}{4}$

Exercise. Find $\mathrm{Arg}\, z$ if $z = 1, i, -1, -i$.

Definition. We write $e^{i\theta} = \cos\theta + i\sin\theta$, so $re^{i\theta} = r\cos\theta + ir\sin\theta$

Remark. ① This notation will be natural if you consider the Taylor Expansion of e^x, $\cos x$, $\sin x$.

② An interesting special case is to take $\theta = \pi$: we get

$e^{i\pi} = \cos\pi + i\sin\pi \Rightarrow e^{i\pi} = 1 + 0$. so we obtain

$$\boxed{e^{i\pi} + 1 = 0}$$

This is the famous Euler's formula, and is regarded as one of the most beautiful formulae in history.

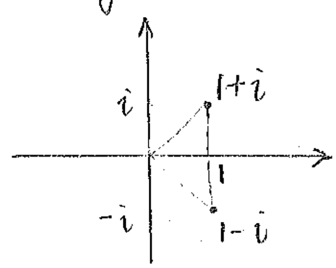
With this notation, we can write a complex number z as

$$z = re^{i\theta}.$$

where $r = |z|$ and $\theta \in \arg z$.

Example. $1+i = \sqrt{2}\, e^{\frac{\pi}{4}i}$

$1-i = \sqrt{2}\, e^{-\frac{\pi}{4}i}$



One advantage of this notation is that it brings a convenient way to compute the product of complex numbers:

Lemma. $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2\, e^{i(\theta_1+\theta_2)}$

Proof. Recall $e^{i\theta} = \cos\theta + i\sin\theta$.

$$\begin{aligned} \text{so } (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \\ &= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned}$$

Example. $(1+i)(1-i) = \sqrt{2}e^{\frac{\pi}{4}i} \cdot \sqrt{2}e^{-\frac{\pi}{4}i} = (\sqrt{2})^2 \cdot e^{(\frac{\pi}{4}-\frac{\pi}{4})i} = 2 \cdot e^{0 \cdot i} = 2(\cos 0 + i\sin 0)$
$= 2$

Corollary. (i) $(re^{i\theta})^{-1} = r^{-1}e^{-i\theta}$

(ii) $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)}$ $\quad (r_2 \neq 0)$.

(iii) $(re^{i\theta})^n = r^n e^{in\theta}$ for any $n \in \mathbb{Z}$.

Proof. (i) $(r^{-1}e^{-i\theta}) \cdot (re^{i\theta}) = (r^{-1} \cdot r)\, e^{i(-\theta+\theta)} = 1$

so $r^{-1}e^{-i\theta} = (re^{i\theta})^{-1}$

(ii) $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})^{-1} = r_1 e^{i\theta_1}\, r_2^{-1} e^{-i\theta_2}$
$= r_1 r_2^{-1} e^{i(\theta_1-\theta_2)}$

(iii) When $n=0$, $(re^{i\theta})^0 = 1$ by default, and $r^0 e^{i0} = 1$.

when $n>0$, we can prove by induction:
① $n=1$: $(re^{i\theta})^1 = r^1 e^{i\theta}$
② Assume $(re^{i\theta})^n = r^n e^{in\theta}$, then

$$\begin{aligned} (re^{i\theta})^{n+1} = (re^{i\theta})^n \cdot (re^{i\theta}) &= r^n e^{in\theta} \cdot re^{i\theta} \\ &= r^{n+1} e^{i(n\theta+\theta)} \\ &= r^{n+1} e^{i(n+1)\theta} \end{aligned}$$

when $n<0$, $(re^{i\theta})^n = [r^{-1}e^{i(-\theta)}]^{-n}$, then apply the previous case to $-n>0$.

Corollary (Moivre's Formula) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Example $(e^{i\theta})^2 = e^{i\cdot 2\theta}$

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$(\cos^2\theta - \sin^2\theta) + i\cdot 2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$, we therefore have

$$\begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{cases}$$

so some of the trigonometric identities we've seen before can be reproved by complex numbers.

The formula $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$ indicates the following:

(1). $|z_1 z_2| = |z_1| \cdot |z_2|$

(2). $\theta_1 \in \arg z_1$, $\theta_2 \in \arg z_2 \Rightarrow \theta_1 + \theta_2 \in \arg(z_1 z_2)$

Example. $\operatorname{Arg} z_1 = \frac{\pi}{3}$, $\operatorname{Arg} z_2 = \frac{3}{4}\pi$. what is $\operatorname{Arg}(z_1 z_2)$?

$\frac{\pi}{3} + \frac{3}{4}\pi = \frac{13}{12}\pi \in \arg(z_1 z_2)$, $\frac{13}{12}\pi - 2\pi = -\frac{11}{12}\pi \in (-\pi, \pi]$.

so $\operatorname{Arg}(z_1 z_2) = -\frac{11}{12}\pi$

n-th Roots of a Complex Number

An interesting application of the exponential form is to use it to compute the n-th roots of a given complex number.

Given a complex number c, we would like to find all complex numbers z such that $z^n = c$, where n is a positive integer.

A first special case is $c=0$: In this case, $z^n = 0$, this implies $|z^n| = 0 \Rightarrow |z|^n = 0 \Rightarrow |z| = 0 \Rightarrow z = 0$ so $z=0$ is the only solution.

A more interesting case is $c=1$: $z^n = 1$
If $z^n = 1$, we say z is an n-th roots of unity.

Recall that $\arg(1) = \{2k\pi \in \mathbb{R} \mid k \in \mathbb{Z}\}$, so if we write $z = re^{i\theta}$, we see $(r \cdot e^{i\theta})^n = e^{2k\pi i}$ for some $k \in \mathbb{Z}$

i.e. $r^n e^{in\theta} = e^{2k\pi i}$

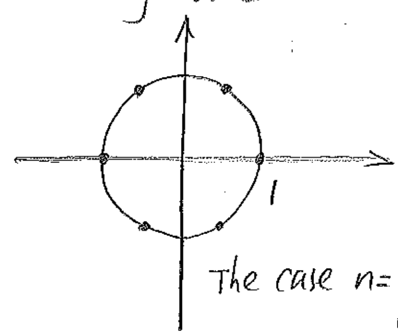
This implies $\begin{cases} r^n = 1, \text{ so } r = 1. \\ n\theta = 2k\pi, \text{ so } \theta = \frac{2k\pi}{n} \end{cases} \Rightarrow z = e^{\frac{2k\pi}{n} i}$ for some $k \in \mathbb{Z}$.

and conversely, if $\theta = \frac{2k\pi}{n}$ for some $k \in \mathbb{Z}$,

$$(e^{i\theta})^n = \left(e^{\frac{2k\pi i}{n}}\right)^n = e^{2k\pi i} = 1.$$

We conclude the set of n-th roots of unity are
$$\left\{ e^{\frac{2k\pi i}{n}} \in \mathbb{C} \mid 0 \le k \le n-1 \right\}$$

Geometrically, they're evenly distributed on the unit circle



The case $n=6$

More generally, we can follow the same idea:

If $C = Re^{i\theta}$ for some $R > 0$, $\theta \in \mathbb{R}$.

The solutions of $z^n = C$ are

$$\left\{ \sqrt[n]{R}\, e^{\frac{\theta + 2k\pi}{n} i} \in \mathbb{C} \,\middle|\, 0 \leq k \leq n-1 \right\}$$

So there're always n solutions to $z^n = C$, where C is a nonzero complex number.

The n solutions are called the n-th roots of C, they're of same norm, and evenly distributed on the circle $|z| = R^{\frac{1}{n}}$.

Example. Solve $z^4 = i$

$$i = e^{(\frac{\pi}{2} + 2k\pi)i}, \; k \in \mathbb{Z}, \quad \text{so} \quad z = e^{\frac{\frac{\pi}{2} + 2k\pi}{4} i},$$

the roots are: $e^{\frac{\pi}{8}i}$, $e^{\frac{5\pi}{8}i}$, $e^{\frac{9\pi}{8}i}$, $e^{\frac{13\pi}{8}i}$

COMPLEX FUNCTIONS

In the subject of Complex Analysis, we are mostly interested in functions $f: \mathbb{C} \longrightarrow \mathbb{C}$, or in some cases the domain is not all $\mathbb{C}$, but a subset of it.

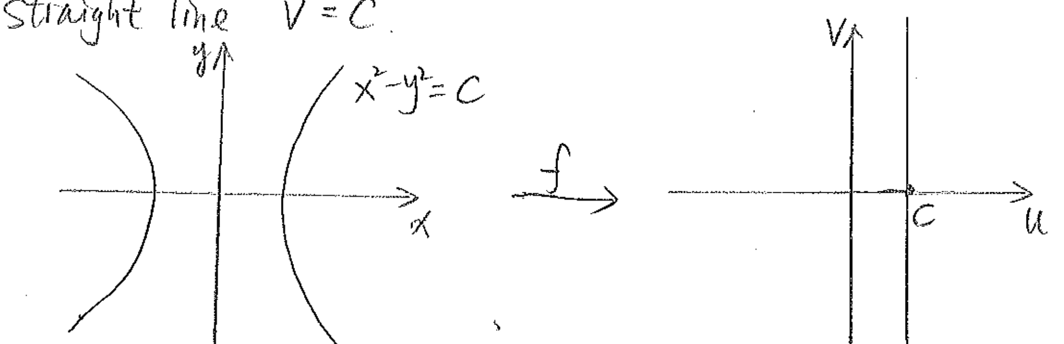
We will take $w = z^2$ as a first example of functions on complex numbers.

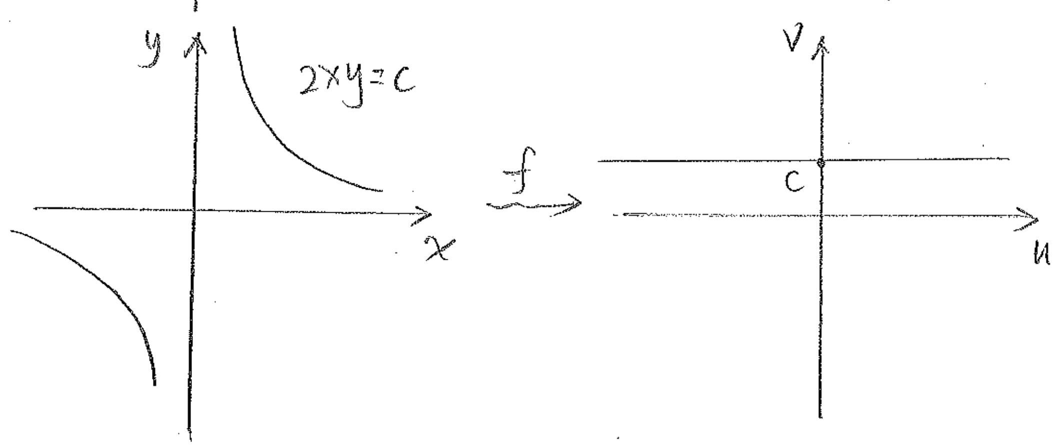
Define $f: \mathbb{C} \longrightarrow \mathbb{C}$, the quadratic function.
$$z \longmapsto z^2$$

If $w = u + vi = f(z) = f(x+yi) = (x+yi)^2 = (x^2-y^2) + 2xyi$

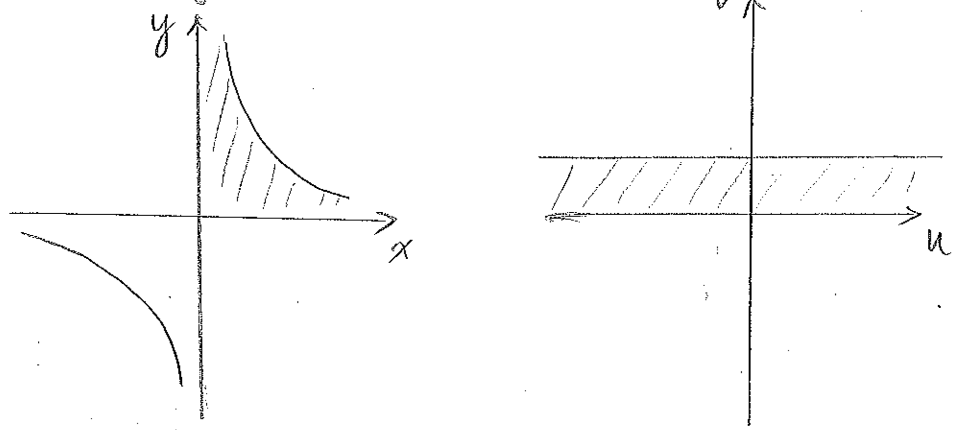
we see $u = x^2 - y^2$ and $v = 2xy$

This implies the function $w = z^2$ sends a curve $x^2 - y^2 = c$ on the complex plane to the straight line $u = c$, and sends the curve $2xy = c$ on the complex plane to the straight line $v = c$.





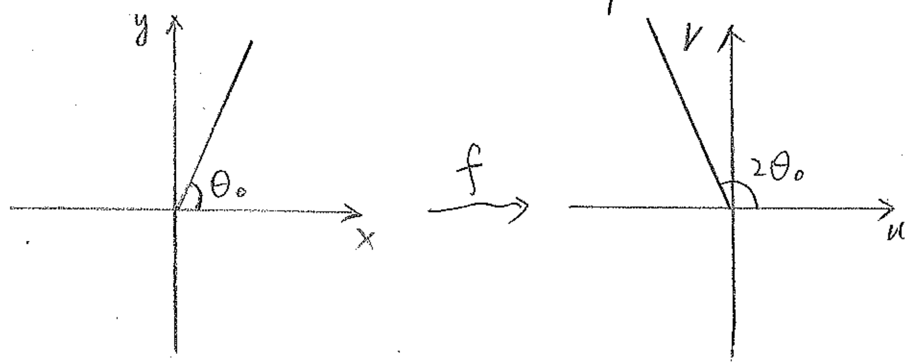
More generally, $w=z^2$ transforms a region in xy-plane to a region in uv-plane:



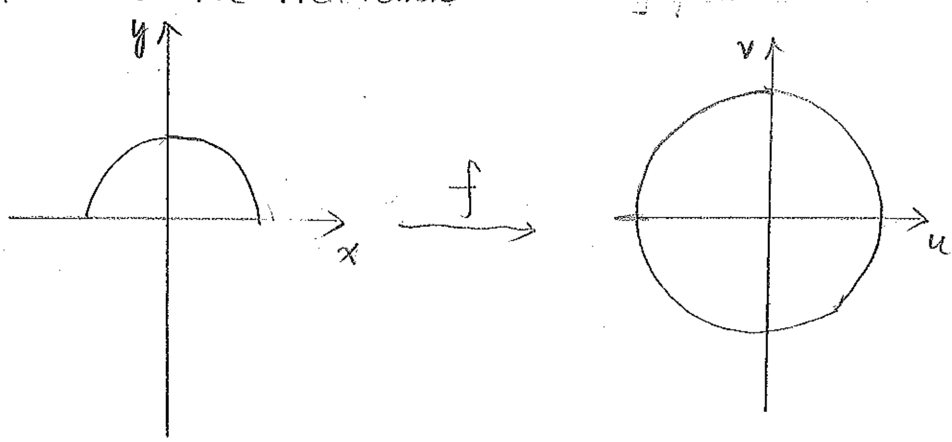
Now if we write the function in the exponential form, we see

$$w = z^2 = (re^{i\theta})^2 = r^2 e^{2\theta i}$$

This indicates each ray $\theta = \theta_0$ on xy-plane is sent to $\theta = 2\theta_0$ on the uv-plane:



and sends the Hemicircle $r=R$, $0 \le \theta \le \pi$ to the circle $r=R^2$.



LIMIT AND CONTINUITY.

We know in Analysis, the foundation of the whole subject is the concept of limit. This definition can be extened to complex functions.

Definition. f is a complex valued function defined at all points z in some deleted neighbourhood of a point $z_0 \in \mathbb{C}$. Define $\lim_{z \to z_0} f(z) = w_0$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon.$$

Intuitively, it means as z approaches z_0, $f(z)$ approaches w_0

Theorem. If the limit $\lim_{z \to z_0} f(z)$ exists, it is unique.

Proof. Suppose $w_0 \neq w_1 \in \mathbb{C}$ are both limits $\lim_{z \to z_0} f(z)$,

i.e. $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$

Let $\varepsilon = \frac{1}{2}|w_0 - w_1| > 0$. there exists $\delta_0 > 0$ and $\delta_1 > 0$

such that $\begin{cases} 0 < |z - z_0| < \delta_0 \Rightarrow |f(z) - w_0| < \varepsilon \\ 0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - w_1| < \varepsilon \end{cases}$

so when $0 < |z - z_0| < \min\{\delta_0, \delta_1\}$.

$$|w_0 - w_1| \leq |w_0 - f(z)| + |f(z) - w_1| < \varepsilon + \varepsilon = 2\varepsilon = |w_0 - w_1|$$

contradiction.

Example. Let $f(z) = \frac{iz}{2}$. We can show that $\lim_{z \to 1} f(z) = \frac{i}{2}$:

For any $\varepsilon > 0$, let $\delta = 2\varepsilon > 0$

For any $0 < |z - 1| < \delta = 2\varepsilon$, $|f(z) - \frac{i}{2}| = |\frac{iz}{2} - \frac{i}{2}| = \frac{1}{2}|z - 1| < \varepsilon$

Example $f(z) = \frac{z}{\bar{z}}$. We can show the limit $\lim_{z\to 0} f(z)$ doesn't exist:

If z approaches 0 from positive real axis,

$$f(z) = \frac{x+0i}{x-0i} = \frac{x}{x} = 1$$

If z approaches 0 from positive imaginary axis,

$$f(z) = \frac{0+yi}{0-yi} = \frac{yi}{-yi} = -1.$$

So we see as z approach 0 from different paths, the $f(z)$ converges to different values. It's impossible for $\lim_{z\to 0} f(z)$ to exist.

Theorem. If $f(z) = f(x+iy) = u(x,y) + v(x,y)i$, then

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \text{ if and only if}$$

$$\lim_{z\to x_0+y_0 i} f(z) = u_0 + v_0 i.$$

Proof. "$\Rightarrow$": For any $\varepsilon > 0$, $\exists \delta_1 > 0$, $\delta_2 > 0$ such that

$$\begin{cases} |(x,y)-(x_0,y_0)| < \delta_1 \Rightarrow |u(x,y)-u_0| < \frac{\varepsilon}{2} \\ |(x,y)-(x_0,y_0)| < \delta_2 \Rightarrow |v(x,y)-v_0| < \frac{\varepsilon}{2} \end{cases}$$

Then for any $|(x+yi)-(x_0+y_0 i)| = \sqrt{(x-x_0)^2+(y-y_0)^2} < \min\{\delta_1, \delta_2\}$

$$\begin{cases} |x-x_0| < \sqrt{(x-x_0)^2+(y-y_0)^2} < \min\{\delta_1,\delta_2\} \\ |y-y_0| < \sqrt{(x-x_0)^2+(y-y_0)^2} < \min\{\delta_1,\delta_2\} \end{cases}$$

So $|u(x,y)-u_0| < \frac{\varepsilon}{2}$, $|v(x,y)-v_0| < \frac{\varepsilon}{2}$

$$|f(x+yi)-(u_0+v_0 i)| = |u(x,y)+v(x,y)i - u_0 - v_0 i| \le |u(x,y)-u_0| + |v(x,y)-v_0| < \varepsilon$$

"$\Leftarrow$": If $\lim_{z\to z_0} f(z) = u_0 + v_0 i$.

Then for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < |(x+yi) - (x_0 + y_0 i)| < \delta \Rightarrow |u(x,y) + v(x,y)i - (u_0 + v_0 i)| < \varepsilon$$

Note: $|(x,y) - (x_0, y_0)| = |(x+yi) - (x_0 + y_0 i)|$.

so $\forall\ 0 < |(x,y) - (x_0, y_0)| < \delta$.

$$|u(x,y) - u_0| \le |(u(x,y) - u_0) + (v(x,y) - v_0)i| < \varepsilon$$

$$|v(x,y) - v_0| \le |(u(x,y) - u_0) + (v(x,y) - v_0)i| < \varepsilon$$

Theorem. If $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} g(z) = w_2$, then

(i) $\lim_{z\to z_0} [f(z) \pm g(z)] = w_1 \pm w_2$.

(ii) $\lim_{z\to z_0} [f(z)\, g(z)] = w_1 w_2$

(iii) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2}$ if $w_2 \neq 0$.

Corollary. If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial in $\mathbb{C}[z]$, then $\lim_{z\to z_0} P(z) = P(z_0)$

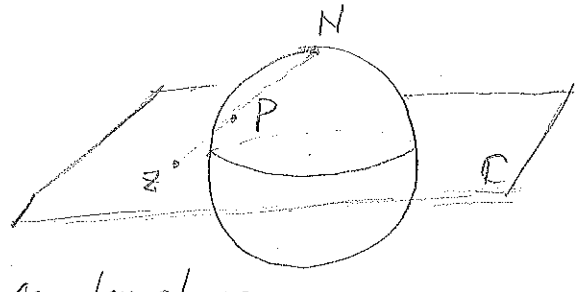
Sometimes we are interested in the behavior of $f(z)$ as $|z| \to \infty$.

First, we can add a "point of infinity" to the complex plane in a very natural way, by constructing the famous Riemann Sphere.

Consider A sphere, with a complex plane passing through its equator, and the point 0 coincides with the center of the sphere

Then for each point P on the sphere not the north pole N. The line passing through N and P intersects the complex plane at a point $z \in \mathbb{C}$. In this way, we can identify the points on the sphere not the north pole with complex numbers, and the north pole then corresponds to ∞.

This indicates the infinity ∞ can also be regarded as an element of complex numbers.



$\mathbb{C} \cup \{\infty\}$ is called the extened complex plane.

The Riemann Sphere motivates the concept of a "neighbourhood" of ∞. For each small $\varepsilon > 0$, the circle $r = \frac{1}{\varepsilon}$ on the complex plane corresponds to a small circle on the Riemann Sphere around the north pole. So $|z| > \frac{1}{\varepsilon}$ is regarded as a neighbourhood of ∞.

Definition. $\lim_{z \to \infty} f(z) = w$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|z| > \delta \Rightarrow |f(z) - w| < \varepsilon$$

Definition $\lim_{z \to z_0} f(z) = \infty$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\varepsilon}$$

Theorem. If z_0 and w_0 are complex numbers, then

(i) $\lim_{z \to z_0} f(z) = \infty$ if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

(ii) $\lim_{z \to \infty} f(z) = w_0$ if $\lim_{z \to z_0} f(\frac{1}{z}) = w_0$

(iii) $\lim_{z \to \infty} f(z) = \infty$ if $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$

Example. $\lim_{z \to -1} \frac{z+1}{iz+3} = 0$, so $\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$.

$$\lim_{z \to \infty} \frac{2z+i}{z+1} = \lim_{z \to 0} \frac{\frac{2}{z}+i}{\frac{1}{z}+1} = \lim_{z \to 0} \frac{2+iz}{1+z} = 2$$

Definition. A function f is continuous at z_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

Proposition. f is continuous at z_0 if and only if:

(i) $\lim_{z \to z_0} f(z)$ exists

(ii) $f(z_0)$ is defined.

(iii) $f(z_0) = \lim_{z \to z_0} f(z)$

Theorem. Compositions of continuous functions results in a continuous function.

Theorem. If a function $f(z)$ is continuous and nonzero at z_0, then $f(z) \neq 0$ throughout some neighbourhood of that point.

The proofs of the above two theorems are similar to their counterpart in calculus / analysis, so we won't discuss about them here. You may try to prove it by recalling the proofs in calculus / analysis

Theorem. Write $f(z) = f(x+yi) = u(x,y) + v(x,y)i$; then

f is continuous at $x_0 + y_0 i$ iff u, v are continuous at (x_0, y_0)

Proof. It follows directly from the Proposition above.

Theorem. If f is continuous throughout a closed and bounded region R on $\mathbb{C}$, then $\max\limits_{z\in R} |f(z)|$ exists.

Proof. Write $f(x+yi) = u(x,y) + v(x,y)i$.

$$|f(x+yi)| = \sqrt{u(x,y)^2 + v(x,y)^2}$$

let $g(x,y) = \sqrt{u(x,y)^2 + v(x,y)^2}$, defined on

$R' = \{(x,y) \in \mathbb{R}^2 \mid x+yi \in R\}$. then by the corresponding result from multi-variable calculus, we know $\max\limits_{R'} g(x,y)$ exists, which is equivalent to $\max\limits_{R} |f(x+yi)|$ exists.

Example. Polynomials are continuous functions on all points of $\mathbb{C}$.

DERIVATIVES & DIFFERENTIATION

Definition. Let f be a complex function whose domain contains a neighbourhood $|z-z_0|<\varepsilon$ of $z_0 \in \mathbb{C}$. Define the derivative of f at z_0 to be

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

If the limit exists, we say f is differentiable at z_0.

Another way of expressing this limit is to write $\Delta z = z - z_0$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

So we can replace z_0 by z in the above expression:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

where $\Delta w = f(z+\Delta z) - f(z)$, if we consider $w = f(z)$.

Example. $f(z) = \frac{1}{z}$. At each nonzero z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{-\Delta z}{(z+\Delta z)z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} -\frac{1}{(z+\Delta z)z}$$

$$= -\frac{1}{z^2}$$

so $f'(z) = \frac{1}{z}$.

Example. $f(z) = \overline{z}$.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

This limit doesn't exists (Recall we have proved a similar limit doesn't exist in previous section). So $f(z)$ is not differentiable at any point

Example. $f(z) = |z|^2 = z \cdot \bar{z}$

$$\frac{\Delta w}{\Delta z} = \frac{(z+\Delta z)(\overline{z+\Delta z}) - z \cdot \bar{z}}{\Delta z} = \frac{z\bar{z} + \Delta z\,\bar{z} + z\overline{\Delta z} + \Delta z \cdot \overline{\Delta z} - z\bar{z}}{\Delta z}$$

$$= \bar{z} + \overline{\Delta z} + z \cdot \frac{\overline{\Delta z}}{\Delta z}$$

If $\Delta z \to 0$ along positive real axis.

$$\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z \longrightarrow \bar{z} + z$$

If $\Delta z \to 0$ along positive imaginary axis.

$$\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z \longrightarrow \bar{z} - z$$

So if $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ exists, $\bar{z} + z = \bar{z} - z$, which implies $z = 0$

it indicates $f(z)$ is NOT differentiable at $z \neq 0$

when $z = 0$. $f'(0) = \lim\limits_{z \to 0} \frac{|z|^2 - |0|^2}{z} = \lim\limits_{z \to 0} \frac{z \cdot \bar{z}}{z} = \lim\limits_{z \to 0} \bar{z} = 0$

So $f(z)$ is only differentiable at $z = 0$.

Example. If f is a complex function such that $f(z) \in \mathbb{R} \subseteq \mathbb{C}$ for all $z \in \mathbb{C}$, and f is differentiable at $z_0 \in \mathbb{C}$, then $f'(z_0) = 0$.

Proof: $f'(z_0) = \lim\limits_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists.

if $\Delta z \to 0$ along real axis. $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \in \mathbb{R}$

if $\Delta z \to 0$ along imaginary axis. $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \in \mathbb{R} i$

So when taking the limit, we see
$f'(z_0) \in \mathbb{R} \cap \mathbb{R} i = \{0\}$

$f'(z_0) = 0$.

BASIC RULES FOR DIFFERENTIATION

Theorem. If f and g are differentiable at $z \in \mathbb{C}$, then $f+g$, $f-g$, $f \cdot g$, $\frac{f}{g}$ (when $g(z) \neq 0$) are all differentiable at z, and:

(i) $(f+g)'(z) = f'(z) + g'(z)$

(ii) $(f-g)'(z) = f'(z) - g'(z)$

(iii) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

(iv) $\left(\frac{f}{g}\right)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

Theorem. If $c \in \mathbb{C}$ is a constant, then

(i) $c' = 0$.

(ii) If f is differentiable at $z \in \mathbb{C}$, then cf is also differentiable at z with $(cf)'(z) = cf'(z)$

Theorem. (Chain Rule) If f has derivative at $z_0 \in \mathbb{C}$, and g has a derivative at $f(z_0) \in \mathbb{C}$, then $F(z) = g(f(z))$ also has derivative at $z_0 \in \mathbb{C}$, with

$$F'(z_0) = g'(f(z_0))f'(z_0)$$

The proofs of the above theorems are almost identical to their counterparts in calculus, so we omit the proofs here.

Example. By induction, we can verify that the power rule still holds for complex function: If n is a nonzero integer, then $(z^n)' = nz^{n-1}$

So for a polynomial $p(z) = C_0 + C_1 z + C_2 z^2 + \cdots + C_n z^n$:

$$p'(z) = C_1 + 2C_2 z + 3C_3 z^2 + \cdots + nC_n z^{n-1}.$$

Example. $f(z) = \dfrac{z-1}{2z+1}$, then

$$f'(z) = \frac{(z-1)'(2z+1) - (z-1)(2z+1)'}{(2z+1)^2}$$

$$= \frac{(2z+1) - 2(z-1)}{(2z+1)^2}$$

$$= \frac{3}{(2z+1)^2}$$

CAUCHY-RIEMANN EQUATIONS

If $f(x+yi) = u(x,y) + v(x,y)i$ has derivative at $z_0 = x_0 + y_0 i$, we would like to find what conditions do $u(x,y), v(x,y)$ need to satisfy.

If $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists,

We may choose to let $\Delta z \to 0$ along real-axis:

$$\lim_{\Delta x \to 0} \frac{f(x_0+y_0 i+\Delta x) - f(x_0+y_0 i)}{\Delta x}$$

$$= \left(\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right) i$$

$$= u_x(x_0, y_0) + v_x(x_0, y_0) i$$

We can also choose to let $\Delta z \to 0$ along imaginary-axis:

$$\lim_{\Delta y \to 0} \frac{f(x_0+y_0 i+\Delta y i) - f(x_0+y_0 i)}{\Delta y i}$$

$$= \left(\lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y\, i}\right) + \left(\lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y\, i}\right) i$$

$$= u_y(x_0, y_0) \cdot \frac{1}{i} + v_y(x_0, y_0)$$

$$= v_y(x_0, y_0) - u_y(x_0, y_0) i$$

We therefore get $u_x(x_0, y_0) + v_x(x_0, y_0) i = v_y(x_0, y_0) - u_y(x_0, y_0) i$

which splits into a pair of equations:

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases}$$

The above equations are called the Cauchy-Riemann Equations, and we can make the conclusion:

Theorem. $f(z) = u(x,y) + v(x,y)i$, and $f'(z)$ exists at $z_0 = x_0 + y_0 i$. Then the first order partial derivatives of u & v exist at (x_0, y_0), satisfying the Cauchy-Riemann Equations, $u_x = v_y$ and $u_y = -v_x$. What's more, we can write

$$f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0)i$$

The above theorem tells us that the Cauchy-Riemann Equations are a necessary condition for f to have derivative at $z_0 \in \mathbb{C}$. But in general, this condition is NOT sufficient:

Example. $f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$

So when $z \neq 0$, $f(z) = \frac{(\bar{z})^2}{z} = \frac{(x - yi)^2}{x + yi}$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + \frac{y^3 - 3x^2y}{x^2 + y^2} i$$

So $u(x,y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ and

$$v(x,y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Then $u_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} = 1$

$$V_y(0,0) = \lim_{\Delta y \to 0} \frac{V(0,\Delta y) - V(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1$$

So $u_x(0,0) = V_y(0,0)$ Similarly we can also verify that $u_y(0,0) = -V_x(0,0)$.

But, f doesn't have derivative at $(0,0)$. It can be proved by taking different paths towards $(0,0)$.

In order to make the Cauchy-Riemann Equations to be sufficient conditions, we need to add more assumptions to the function:

Theorem. $f(z) = u(x,y) + v(x,y)i$ is defined throughout some neighbourhood of $z_0 = x_0 + y_0 i$, and suppose:

(i) u_x, u_y, V_x, V_y exist throughout the neighbourhood.

(ii) u_x, u_y, V_x, V_y are continuous at (x_0, y_0)

(iii) $\begin{cases} u_x(0,0) = V_y(0,0) \\ u_y(0,0) = -V_x(0,0) \end{cases}$

Then $f'(z_0)$ exists, and $f'(z_0) = u_x(x_0, y_0) + V_x(x_0, y_0) i$

Proof. Part (i) and (ii) together imply u and v are differentiable functions (Recall that for multivariable functions, differentiability and existence of partial derivatives are not equivalent)

Since u and v are differentiable in the neighbourhood,

$$\Delta u = u(x,y) - u(x_0,y_0) = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Delta v = v(x,y) - v(x_0,y_0) = v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y.$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

We then get

$$\begin{aligned} f(z_0+\Delta z) - f(z_0) &= \Delta u + \Delta v \cdot i \\ &= u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2 \Delta y \\ &\quad + [v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y]\, i \\ &= u_x(x_0,y_0)\Delta x - v_x(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\ &\quad + [v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y]\, i \\ &= u_x(x_0,y_0)\Delta z + v_x(x_0,y_0)\, i \cdot \Delta z \\ &\quad + (\varepsilon_1+\varepsilon_3)\Delta x + (\varepsilon_2+\varepsilon_4)\Delta y \end{aligned}$$

So
$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = u_x(x_0,y_0) + v_x(x_0,y_0)\, i + (\varepsilon_1+\varepsilon_3)\cdot\frac{\Delta x}{\Delta z} + (\varepsilon_2+\varepsilon_4)\cdot\frac{\Delta y}{\Delta z}$$

as $\Delta z \to 0$, we see

$$\lim_{z\to z_0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = u_x(x_0,y_0) + v_x(x_0,y_0)\, i$$

So $f'(z_0) = u_x(x_0,y_0) + v_x(x_0,y_0)\, i$

Example. $f(z) = |z|^2 = (x^2+y^2) + 0i$

So $u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$.

The Cauchy-Riemann equations are not satisfied unless $(x,y) = (0,0)$, so $f(z)$ is not differentiable at $z \neq 0$.

When $(x,y) = (0,0)$ we see u and v have continuous partial derivatives in a neighbourhood of $(0,0)$, and the Cauchy-Riemann Equations are satisfied, so we can conclude $f'(z)$ exists at $z=0$

Example. $f(z) = x^3 + i(1-y)^3$

So $u(x,y) = x^3$, $v(x,y) = (1-y)^3$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Leftrightarrow \begin{cases} 3x^2 = -3(1-y)^2 \\ 0 = 0 \end{cases} \Leftrightarrow x^2 + (y-1)^2 = 0$$

$$\Leftrightarrow (x,y) = (0,1)$$

We see if $(x,y) \neq (0,1)$, the Cauchy-Riemann Equations are not satisfied, so $f'(z)$ doesn't exist if $z \neq i$.

If $(x,y) = (0,1)$, the Cauchy-Riemann Equations are satisfied, and u_x, u_y, v_x, v_y are continuous, so $f'(i)$ exists and $f'(i) = u_x(0,1) + v_x(0,1)i = 0$

ANALYTIC FUNCTIONS

Definition. A complex function f is analytic in an open set S if it has derivative everywhere in S. f is analytic at $z_0 \in \mathbb{C}$ if its analytic in some neighbourhood of z_0.

Definition. An entire function is a function that is analytic at each point of $\mathbb{C}$.

Example
- $f(z) = z^2$ is an entire function
- $f(z) = \frac{1}{z}$ is analytic at any nonzero point.
- $f(z) = |z|^2$ is NOT analytic anywhere

The basic differentiation rules indicate the following:

Proposition ① If f and g are analytic functions on an open connected set D, then $f \pm g$, $f \cdot g$ are analytic on D, and $\frac{f}{g}$ is analytic in D provided g doesn't vanish on D

② If f is analytic on an open connected set D, and $f(D)$ is contained in an open connected set on which g is analytic, then $g \circ f$ is analytic on D.

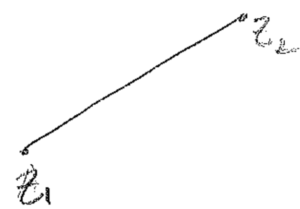
Definition
- An open set $S \subseteq \mathbb{C}$ is connected if any pair of elements in S can be connected by a polygonal line in S, consisting of finitely number of segments.
- A non-empty open connected set is called a domain
- A domain with none, some, or all its boundary points is called a region

Proposition. If $f'(z)=0$ everywhere in a domain D, then f is a constant function on D

Proof $f'(z) = U_x + V_x i = 0$ and $U_x = V_y$, $U_y = V_x$.

So $U_x = U_y = V_x = V_y = 0$ on D

We only need to show f is constant on any line segment in D, then since D is a domain, any two points are connected, we can therefore conclude f is constant throughout D.



We can parameterize a line segment by $(x(t), y(t))$.

So $\frac{d}{dt}U(x(t), y(t)) = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = 0$

it indicates $U(x(t), y(t))$ is a constant function.
So U is constant on the line segment.
Similarly, V is also constant

Corollary. If $f(z)$ and $\overline{f(z)}$ are both analytic inside a domain D, then $f(z) \equiv C \in \mathbb{C}$ on D.

Proof. If $f(z) = U(x,y) + V(x,y)i$, then $\overline{f(z)} = U(x,y) - V(x,y)i$

$f(z)$ is analytic on $D \Rightarrow \begin{cases} U_x = V_y \\ U_y = -V_x \end{cases}$ on D.

$\overline{f(z)}$ is analytic on $D \Rightarrow \begin{cases} U_x = (-V)_y = -V_y \\ U_y = -(-V)_x = V_x \end{cases}$ on D

So we get $u_x = u_y = v_x = v_y$ on D

which implies $f'(z) = u_x + v_x i = 0$ on D.

by the previous propositon, $f(z) \equiv C \in \mathbb{C}$ on D.

Corollary. If $f(z)$ is an analytic function on a domain D, with $|f(z)| \equiv r \in \mathbb{R}^{\geq 0}$ on D, then $f(z) \equiv C \in \mathbb{C}$ on D

Proof. If $r = 0$, then $|f(z)| \equiv 0 \Rightarrow f(z) \equiv 0$.

If $r > 0$. then $f(z) \cdot \overline{f(z)} = |f(z)|^2 = r^2$

so $\overline{f(z)} = \dfrac{r^2}{f(z)}$ is analytic, since

$f(z)$ is analytic.

By the previous Corollary, $f(z), \overline{f(z)}$ both analytic on $D \Rightarrow f(z) \equiv C \in \mathbb{C}$ on D.

Definition. $H(x,y): \mathbb{R}^2 \supseteq D \longrightarrow \mathbb{R}$ is called harmonic on D if

$$H_{xx} + H_{yy} = 0 \quad \text{throughout } D.$$

and H has continuous first and second order partial derivatives.

Theorem. If $f(z) = u(x,y) + v(x,y)i$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof. $f(z)$ is analytic $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{cases}$

So $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$.

Similarly we can prove $v_{xx} = v_{yy}$.

Remark. In the above proof, we do need to verify u and v have continuous first and second order derivatives, but the proof of this needs the theorem that

$f(z)$ is analytic $\Rightarrow$ $f(z)$ has derivatives of arbitrary order.

which shall be proved later in this course.

Example. $f(z)=\frac{1}{z^2}$ is analytic at any $z\neq 0$,

$$f(z)=\frac{1}{z^2}=\frac{1}{z^2}\cdot\frac{\bar{z}^2}{\bar{z}^2}=\frac{\bar{z}^2}{|z|^4}=\frac{(x^2-y^2)-2xyi}{(x^2+y^2)^2}$$

So $u(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$ and $v(x,y)=\dfrac{-2xy}{(x^2+y^2)^2}$ are harmonic functions, on any domain not containing origin

Remark. Harmonic functions play an important role in analysis and applied math. For functions with n variables $u(x_1,\dots,x_n)$. we define u to be harmonic if

$$u_{x_1x_1}+u_{x_2x_2}+\cdots+u_{x_nx_n}=0$$

Definition. If $u(x,y)$ is harmonic, we define $v(x,y)$ to be a harmonic conjugate of $u(x,y)$ if $f(z)=u(x,y)+v(x,y)i$ is analytic.

So in the above example, we see $v(x,y)=\dfrac{-2xy}{(x^2+y^2)^2}$ is a harmonic conjugate of $u(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$

Exercise. Show that the harmonic conjugate of $u(x,y)$ on a domain D is unique up to adding a constant.

THE EXPONENTIAL FUNCTIONS

Definition. The exponential function $f(z)=e^z$ is defined by

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

Proposition (i) If $z=x+yi \in \mathbb{C}$, then $|e^z| = e^x$

(ii) $e^{z_1}e^{z_2} = e^{z_1+z_2} \quad \forall\, z_1, z_2 \in \mathbb{C}$

(iii) $f(z)=e^z$ is an entire function, and $f(z)=e^z$

Proof. (i) $|e^z| = |e^x \cdot e^{iy}| = |e^x|\cdot|e^{iy}| = e^x \cdot 1 = e^x$

(ii) If $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$,

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}e^{y_1 i}e^{x_2}e^{y_2 i} \\ &= (e^{x_1}e^{x_2})(e^{y_1 i}e^{y_2 i}) \\ &= e^{x_1+x_2}\, e^{(y_1+y_2)i} \\ &= e^{z_1+z_2} \end{aligned}$$

(iii) $f(z) = e^z = e^x\cos y + ie^x\sin y$

So $u(x,y) = e^x\cos y$, $v(x,y) = e^x\sin y$

$$\begin{cases} u_x = e^x\cos y = v_y \\ u_y = -e^x\sin y = -v_x \end{cases}$$

We see u_x, u_y, v_x, v_y are all continuous, satisfying Cauchy-Riemann Equations everywhere, so $f(z)=e^z$ is entire.

$$\begin{aligned} f'(z) &= u_x(x,y) + v_x(x,y)i \\ &= e^x\cos y + ie^x\sin y \\ &= e^z \end{aligned}$$

Remark The complex exponential function has some properties that the real exponential function doesn't have:

① $f(z)=e^z$ has period $2\pi i$.

② $f(z)=e^z$ may be a negative real number.

For example. $f(\pi i)=e^{\pi i}=-1$

More generally, we have the following proposition.

Proposition (i) $\mathrm{Range}(e^z)=\mathbb{C}\setminus\{0\}$

(ii) If $e^{z_1}=e^{z_2}$, then $z_1-z_2=2\pi i\cdot k$ for some $k\in\mathbb{Z}$

Proof. (i) Given any complex number $z_0\neq 0$, we can write it in the form $z_0=re^{i\theta}$ for some $r>0$ and $\theta\in\mathbb{R}$.

Since $r>0$, there $\exists\ x\in\mathbb{R}$ such that $r=e^x$.

$z_0=re^{i\theta}=e^x\cdot e^{i\theta}=e^{x+i\theta}=f(x+i\theta)$

Also, $e^z\neq 0$ for any $z\in\mathbb{C}$ since $|e^z|=|e^x|>0$.

(ii). Write $z_1=x_1+y_1 i$, $z_2=x_2+y_2 i$.

$e^{z_1}=e^{z_2}\Rightarrow e^{x_1}\cdot e^{y_1 i}=e^{x_2}\cdot e^{y_2 i}$

$|e^{z_1}|=|e^{z_2}|\Rightarrow e^{x_1}=e^{x_2}\Rightarrow x_1=x_2$.

So $e^{y_1 i}=e^{y_2 i}\Rightarrow e^{(y_1-y_2)i}=1\Rightarrow y_1-y_2=2\pi i k$ for some $k\in\mathbb{Z}$

THE LOGARITHMIC FUNCTION

Recall that in the previous section, we described that given $z \in \mathbb{C} \setminus \{0\}$, how to find $w \in \mathbb{C}$ such that $e^w = z$, and different solutions are differed by $2\pi i k$, $k \in \mathbb{Z}$.

More concretely, if we write $w = w_1 + w_2 i$, and $z = |z| e^{i\theta}$, then $e^w = z$ means $e^{w_1} \cdot e^{w_2 i} = |z| \cdot e^{i\theta}$.

$$\text{So } \begin{cases} e^{w_1} = |z| \\ e^{w_2 i} = e^{i\theta} \end{cases} \Rightarrow \begin{cases} w_1 = \ln|z| \\ w_2 = \theta + 2k\pi \in \arg(z) \end{cases}$$

We conclude w can be taken to be $w = \ln|z| + i\arg(z)$

Following the idea that logrithmic function should be the inverse of exponential function, we can define

$$f(z) = \log z = \ln|z| + i\arg(z) \quad (z \neq 0)$$

Note that this is not the "function" we usually refer to, and instead, it's a "multi-value function".

One way to make it into a single value function is to make a choice of $\arg(z)$ for each $z \in \mathbb{C} \setminus \{0\}$, and a natural choice is the principal argument $\operatorname{Arg}(z)$

Definition. If $z \neq 0$, the pincipal value of $\log z$ is $\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z)$

Remark. When restricted to positive real numbers, $\operatorname{Log} z$ agress with the real logarithmic function.

Example. $\log(-1) = \log|-1| + i\arg(-1) = (\pi + 2k\pi)i$. $k \in \mathbb{Z}$.

$\operatorname{Log}(-1) = \pi i$.

Example. Verify $\text{Log}[(1+i)^2] = 2\,\text{Log}(1+i)$

$$\text{Log}[(1+i)^2] = \text{Log}(2i) = \ln|2i| + i\,\text{Arg}(2i) = \ln 2 + \frac{\pi}{2} i$$

$$2\,\text{Log}(1+i) = 2\left(\ln|1+i| + i\,\text{Arg}(1+i)\right) = 2\left(\ln\sqrt{2} + \frac{\pi}{4} i\right)$$
$$= \ln 2 + \frac{\pi}{2} i$$

Example. Verify $\log(i^2) \neq 2\log i$

$$\log i^2 = \log(-1) = \ln|-1| + i\arg(-1) = (\pi + 2k\pi)i, \; k \in \mathbb{Z}.$$

$$2\log i = 2\left(\ln|i| + i\arg(i)\right) = 2\, i\left(\frac{\pi}{2} + 2k\pi\right) = (\pi + 4k\pi)i, \; k \in \mathbb{Z}$$

Example. $\text{Log}(-1)^2 = \text{Log}\, 1 = 0, \quad 2\,\text{Log}(-1) = 2(i\pi) = 2\pi i$

So $\text{Log}(-1)^2 \neq 2\,\text{Log}(-1)$

We now want one more step further: we wish to have a single value logarithmic function that is continuous:

Let $\alpha \in \mathbb{R}$, we delete the ray $\theta = \alpha$ from the domain, and define
$$\log(z) = \ln|z| + i\theta, \quad \text{where } \theta = \arg(z) \cap (\alpha, \alpha + 2\pi)$$

Then the function we obtain is continuous on this smaller domain since both real and imaginary parts are continuous now.

In your homework, you have checked the polar form of the Cauchy-Riemann Equations are $\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$

$\log(re^{i\theta}) = \ln r + i\theta$, so $\begin{cases} r u_r = r \cdot \frac{1}{r} = 1 = v_\theta \\ u_\theta = 0 = -r \cdot 0 = -r v_r \end{cases}$

So this function is analytic.

Next, we are going to use the expression of $f'(z)$ in the polar form: $f'(z) = e^{-i\theta}(u_r + iv_r)$

So for $f(z) = \log z$,

$$f'(z) = e^{-i\theta}\left(\frac{\partial \ln r}{\partial r} + r\frac{\partial \theta}{\partial r}\right)$$

$$= e^{-i\theta} \cdot \frac{1}{r}$$

$$= \frac{1}{re^{i\theta}}$$

$$= \frac{1}{z}$$

we thus get : $(\log z)' = \frac{1}{z}$

Definition A branch of a multi-value function f is a single value function F that is analytic in some domain at each z of which $F(z)$ is one of the values of $f(z)$.

When applied to $\log z$, each $\log z = \ln|z| + i\theta$. $(\alpha < \theta < \alpha + 2\pi)$ is a branch, and the function $\operatorname{Log} z = \ln|z| + i\theta$. $(-\pi < \theta < \pi)$ is called the principal branch.

Definition
- A branch cut is a portion of a line or curve that is removed from $\mathbb{C}$ in order to define a branch F of a multi-value function f.
- Points on the branch cut are called singular points for F.
- A point that is common to all branch cuts of f is called a branch point

Example For $\operatorname{Log} z = \ln|z| + i\theta$ $(-\pi < \theta < \pi)$ the ray $\theta = \pi$ is the branch cut, and 0 is a branch point for the multi-value function $\log z$

Remark. We can also obtain the derivative of a branch $f(z) = \log z = \ln r + i\theta$, $(\alpha < \theta < \alpha + 2\pi)$ by the Chain Rule.

First, $e^{\log z} = z$, so we can differentiate both sides using the Chain Rule.

$$\frac{d}{dz}(e^{\log z}) = \frac{d}{dz}(z)$$

$$e^{\log z} \cdot (\log z)' = 1$$

$$z \cdot (\log z)' = 1$$

$$\log z = \frac{1}{z}$$

By making a branch cut, we can obtain an analytic branch of the $\log z$ multi-valued function, but there're also properties that are only true when we regard $\log z$ as a multi-valued function.

Example. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$.

This is because

$$\begin{aligned}\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2) \\ &= \ln|z_1| + \ln|z_2| + i(\arg(z_1) + \arg(z_2)) \\ &= \ln|z_1| + i\arg(z_1) + \ln|z_2| + i\arg(z_2) \\ &= \log(z_1) + \log(z_2).\end{aligned}$$

(Note $\arg(z)$ is a set instead of a number!)

But if we take a branch cut, $\log(z) = \ln|z| + i\theta$, $(0 < \theta < 2\pi)$, then $\log((-i)\cdot(-i)) = \log(-1) = \pi i$, while $\log(-i) + \log(-i) = \frac{3}{2}\pi i + \frac{3}{2}\pi i = 3\pi i$.

THE POWER FUNCTION

We are going to study functions of the form $f(z) = z^c$, where $c \in \mathbb{C}$ is a constant.

First, we know what $f(z) = z^n$ is when $n \in \mathbb{N}$:

$$f(z) = z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n \text{ copies of } z}$$

An important observation which will be useful later is that

$$z^n = e^{n \log z}.$$

This can be verified by:

$$\begin{aligned} e^{n \log z} = e^{n(\ln|z| + i \arg(z))} &= e^{n \ln|z|} \cdot e^{i n \arg(z)} \\ &= e^{\ln|z|^n} \cdot e^{i n \arg(z)} \\ &= |z|^n \cdot e^{i n \arg(z)} \\ &= z^n \end{aligned}$$

Next, let's think about the function $f(z) = z^{\frac{1}{n}}$, $n \in \mathbb{N}$.
Recall that we've discussed about this kind of function before.
If $w = z^{\frac{1}{n}}$, there're n solutions for w when $z \neq 0$.

i.e. $w^n = z = r e^{i \arg(z)}$

so $w = \sqrt[n]{r}\, e^{i \frac{1}{n} \arg(z)}$

Observe
$$\begin{aligned} \frac{1}{n} \arg(z) &= \frac{1}{n} \{ \operatorname{Arg}(z) + 2k\pi \in \mathbb{R} \mid k \in \mathbb{Z} \} \\ &= \left\{ \frac{\operatorname{Arg}(z)}{n} + \frac{2k\pi}{n} \in \mathbb{R} \mid k \in \mathbb{Z} \right\} \\ &= \left\{ \frac{\operatorname{Arg}(z)}{n} + \frac{2k\pi}{n} \in \mathbb{R} \mid k \in \mathbb{Z} \cap [0, n-1] \right\} \end{aligned}$$

This implies

$$f(z) = z^{\frac{1}{n}} = \left\{ |z|^{\frac{1}{n}} \cdot e^{i\theta} \in \mathbb{C} \;\middle|\; \theta = \frac{Arg(z)}{n} + \frac{2k\pi}{n},\ 0 \le k \le n-1 \right\}$$

Then observe $z^{\frac{1}{n}} = e^{\frac{1}{n}\log z}$ is also true:

$$e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + i\arg(z))} = e^{\frac{1}{n}\ln|z|} \cdot e^{i\frac{\arg(z)}{n}}$$

$$= \left\{ |z|^{\frac{1}{n}} \cdot e^{i\theta} \in \mathbb{C} \;\middle|\; \theta = \frac{Arg(z) + 2k\pi}{n},\ 0 \le k \le n-1 \right\}$$

$$= z^{\frac{1}{n}}$$

The above observations motivates the definition:

Definition | If $c \in \mathbb{C}$, the power function $f(z) = z^c$ is defined by

$$z^c = e^{c\log z} \quad (z \neq 0)$$

Remark | In general, it'll be a multi-valued function.

Example | $(\sqrt{2})^i = e^{i\log\sqrt{2}} = e^{i(\ln\sqrt{2} + i\cdot 2\pi k)} = e^{-2\pi k + i\ln\sqrt{2}}$, $k \in \mathbb{Z}$.

So there're infinitely many values of $(\sqrt{2})^i$

Proposition | $(z^c)^{-1} = z^{-c}$ (We leave the proof as homework)

Similar to the case of $\log z$, we can make z^c a single valued function by choosing a branch (making a branch cut). Since $z^c = e^{c\log z}$, once we choose a branch for $\log z$, $\log z$ will be single-valued, therefore z^c will also be single-valued.

For a branch of $\log z$, it's analytic, and exponential function is also analytic, the composition $z^n = e^{c\log z}$ is also analytic on a branch $\alpha < \theta < \alpha + 2\pi$:

$$\frac{d}{dz}(z^c) = \frac{d}{dz} e^{c\log z} = e^{c\log z} \cdot (c\log z)' = e^{c\log z} \cdot \frac{c}{z}$$

$$= c \cdot \frac{e^{c\log z}}{e^{\log z}}$$

$$= c\, e^{(c-1)\log z}$$

$$= c\, z^{c-1}$$

And similar as the $\log(z)$ function, we take $-\pi < \theta < \pi$ as the principal branch of z^c.

THE TRIGONOMETRIC FUNCTIONS

We have defined $e^{ix} = \cos x + i\sin x$ for any real $x \in \mathbb{R}$

Replace x by $-x$, we get $e^{-ix} = \cos(-x) + i\sin(-x)$
$= \cos x - i\sin x$

So we have a system of two equations

$$\begin{cases} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{cases}$$

which implies $\begin{cases} \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \\ \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} \end{cases}$

Since we have extended the exponential function to $\mathbb{C}$, it's natural now to define the sine and cosine functions:

$$\begin{cases} \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \\ \cos z = \dfrac{e^{iz} + e^{-iz}}{2} \end{cases} \qquad (z \in \mathbb{C})$$

Example $\sin \pi i = \dfrac{e^{i(\pi i)} - e^{-i(\pi i)}}{2i} = \dfrac{e^{-\pi} - e^{\pi}}{2i}$

$\cos \pi i = \dfrac{e^{i(\pi i)} + e^{-i(\pi i)}}{2} = \dfrac{e^{-1} + e}{2}$

Proposition (i) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

(ii) $\sin^2 z + \cos^2 z = 1$.

The proofs are left as exercises.

Proposition $\sin z$ and $\cos z$ are entire functions, with

$(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$

Proof They're entire since they're linear combinations of the entire functions e^{iz} & e^{-iz}

Then apply the chain rule to find their derivatives

Proposition $\sin z$ and $\cos z$ both have period 2π

Proof $$\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{iz}\cdot e^{2\pi i} - e^{-iz}\cdot e^{-2\pi i}}{2i} = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos(z+2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{iz}\cdot e^{2\pi i} + e^{-iz}\cdot e^{-2\pi i}}{2} = \frac{e^{iz}+e^{-iz}}{2}$$

Remark. $\sin z$ and $\cos z$ are NOT bounded, which is different from the real case.

For example, $\cos(Ni) = \dfrac{e^{i(Ni)} + e^{-i(Ni)}}{2} = \dfrac{e^{-N}+e^{N}}{2}$

We see $\lim\limits_{N\to+\infty} \cos(Ni) = \infty$

Definition A zero of a function f is a number z_0 such that $f(z_0)=0$

Example The real function $f(x) = x^2+1$ has no zeros

The complex function $f(z) = z^2+1$ has $\pm i$ as zeros

Proposition. All the zeros of $\sin z$ and $\cos z$ are real.

Proof $$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{1}{2i}\left[e^{ix}\cdot e^{-y} - e^{-ix}\cdot e^{y}\right]$$

We see $\sin z = 0$ iff. $e^{ix} \cdot e^{-y} = e^{-ix} \cdot e^{y}$

iff. $e^{2ix} = e^{2y}$

Note $|e^{i \cdot 2x}| = 1$, so $e^{2ix} = e^{2y} \Rightarrow e^{2y} = 1 \Rightarrow y = 0$

which further implies $e^{i \cdot 2x} = 1 \Rightarrow 2x = 2k\pi$, $k \in \mathbb{Z}$.

$\Rightarrow x = k\pi$, $k \in \mathbb{Z}$

We see the only zeros of $\sin z$ are $k\pi$, $k \in \mathbb{Z}$.

Similarly, we can prove that the zeros of $\cos z$ are $\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

Definition. $\tan z = \frac{\sin z}{\cos z}$ $\quad$ $\cot z = \frac{\cos z}{\sin z}$

$\sec z = \frac{1}{\cos z}$ $\quad$ $\csc z = \frac{1}{\sin z}$

Definition. If f is not analytic at z_0, but is analytic at some point in every neighbourhood of z_0, then z_0 is called a singular point, or singularity of f.

By this definition, we see the zeros of $\cos z$ are the singularities of $\tan z$ and $\sec z$, and the zeros of $\sin z$ are the singularities of $\cot z$ and $\csc z$

DEFINITE INTEGRAL

We are going to consider a function $w: I \longrightarrow \mathbb{C}$ given by $w(t) = u(t) + v(t)i$, where $I = [a, b]$ is some interval on $\mathbb{R}$, and $u(t)$, $v(t)$ are real-valued functions on I

Definition. If the derivatives $u'(t)$ and $v'(t)$ exist, define the derivative $w'(t) = u'(t) + v'(t)i$

Example. $w(t) = e^{z_0 t}$, where $z_0 = x_0 + y_0 i$ is a constant.

$$w(t) = e^{z_0 t} = e^{x_0 t} \cdot e^{i y_0 t}$$
$$= e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t$$

So $w'(t) = (e^{x_0 t} \cos y_0 t)' + (i e^{x_0 t} \sin y_0 t)'$

$$= x_0 e^{x_0 t} \cos y_0 t - y_0 e^{x_0 t} \sin y_0 t + x_0 i e^{x_0 t} \sin y_0 t + y_0 i e^{x_0 t} \cos y_0 t$$
$$= z_0 e^{x_0 t} \cos y_0 t + z_0 i e^{x_0 t} \sin y_0 t$$
$$= z_0 e^{x_0 t} \cdot e^{i y_0 t}$$
$$= z_0 e^{(x_0 + i y_0) t}$$
$$= z_0 e^{z_0 t}$$

Proposition. $w: I \longrightarrow \mathbb{C}$ and $f: \mathbb{C} \longrightarrow \mathbb{C}$, If $\frac{dw}{dt}(t)$ and $\frac{df}{dz}(w(t))$ both exist, then $\frac{d f\circ w}{dt}(t)$ exists and $\frac{d f\circ w}{dt}(t) = \frac{df}{dz}(w(t)) \cdot \frac{dw}{dt}(t)$

Write $w(t) = u(t) + v(t)i$,

$$f(z) = f(x + yi) = U(x, y) + V(x, y)i$$

Then $f \circ w(t) = U(u(t), v(t)) + V(u(t), v(t))i$

$$\frac{d}{dt}(f\circ w) = \frac{d}{dt}U(u(t),v(t)) + i\cdot\frac{d}{dt}V(u(t),v(t))$$

$$= \frac{\partial U}{\partial x}\cdot\frac{du}{dt} + \frac{\partial U}{\partial y}\cdot\frac{dv}{dt} + i\frac{\partial V}{\partial x}\cdot\frac{du}{dt} + i\cdot\frac{\partial V}{\partial y}\cdot\frac{dv}{dt}$$

$$= \left(\frac{\partial U}{\partial x} + i\cdot\frac{\partial V}{\partial x}\right)\frac{du}{dt} + \left(\frac{\partial U}{\partial y} + i\frac{\partial V}{\partial y}\right)\frac{dv}{dt}$$

$$= \frac{df}{dz}\cdot\frac{du}{dt} + i\left(\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}i\right)\frac{dv}{dt}$$

$$= \frac{df}{dz}\frac{du}{dt} + i\cdot\frac{df}{\partial z}\frac{dv}{dt}$$

$$= \frac{df}{dz}\left(\frac{du}{dt} + i\frac{dv}{dt}\right)$$

$$= \frac{df}{dz}\cdot\frac{dw}{dt}$$

Example. If $w(t): I \to \mathbb{C}$ has derivative, then $\frac{d}{dt}w(t)^2 = 2w(t)\cdot w'(t)$

Definition. If $w(t) = u(t) + v(t)i$, define the definite integral of $w(t)$ over an interval $[a,b]$ to be

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

provided the integrals $\int_a^b u(t)\,dt$, $\int_a^b v(t)\,dt$ exist.

Example. $$\int_0^{\frac{\pi}{4}} e^{it}\,dt = \int_0^{\frac{\pi}{4}}(\cos t + i\sin t)\,dt = \int_0^{\frac{\pi}{4}}\cos t\,dt + i\int_0^{\frac{\pi}{4}}\sin t\,dt$$

$$= \frac{\sqrt{2}}{2} + i\left(1 - \frac{\sqrt{2}}{2}\right)$$

Example. The Fundamental Theorem of Calculus can also be extended to this definition:

If $W'(t) = w(t)$, then $\int_a^b w(t)\,dt = W(b) - W(a)$

Its proof is straight-forward, just consider each of the real part and the imaginary part :

If $W(t) = U(t) + V(t)\,i$

$w(t) = u(t) + v(t)\,i$,

and $W'(t) = U'(t) + V'(t)\,i = w(t)$

we see $U'(t) = u(t)$, $V'(t) = v(t)$

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$$= U(b) - U(a) + i\left(V(b) - V(a)\right)$$

$$= W(b) - W(a)$$

Proposition. If $w(t) = u(t) + iv(t)$, and $u(t)$, $v(t)$ are piecewise continuous on $[a,b]$, then $\int_a^b w(t)\,dt$ exists.

CONTOUR INTEGRALS.

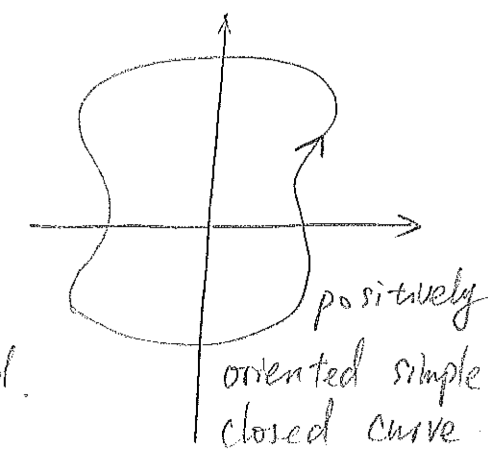
Definition. An arc in the complex plane is a continuous function $[a,b] \longrightarrow \mathbb{C}$ $z(t) = x(t) + y(t)i$, with image points ordered according to increasing values of t.

Example. $z(t) = t + t^2 i$, $t \in [0,1]$ is a part of a parabola.

Definition. An arc C is simple if it has no self-intersection.
i.e. $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$
An arc C is closed if $z(a) = z(b)$.
C is a simple closed curve, or a Jordan curve if it's simple and closed. It's positively oriented if it's in counterclockwise direction.

Example The unit circle $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ is a simple closed curve, positively oriented.

$z = e^{-i\theta}$, $0 \le \theta \le 2\pi$ is a simple closed curve, negatively oriented.



Note: Arcs are functions, so they may still be different arcs even if their images are the same geometric figure.

For example $z = e^{i2\theta}$ $(0 \le \theta \le 2\pi)$ is a different arc from $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$

But at the same time, the same arc may admit different parametrizations:

If $z(t)$ is a curve defined on $t \in [a,b]$ and $t = \phi(\tau)$ is a strictly increasing continuously differentiable on $[\alpha, \beta]$ with $\phi(\alpha) = a$, $\phi(\beta) = b$ then $Z(\tau) = z(\phi(\tau))$ $\alpha \le \tau \le \beta$ represents the same arc as $z(t)$, $a \le t \le b$.

Example $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ and $z = e^{i2\alpha}$, $0 \le \alpha \le \pi$ represent the same arc.

Definition. If an arc is given by $z(t) = x(t) + y(t)i$, we say $z(t)$ is differentiable if $z'(t)$ exists.

Recall that the arclength of an arc on $\mathbb{R}^2$ is defined to be $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$, we see the arclength of the arc $z(t)$ $a \le t \le b$ is computed by $\int_a^b |z'(t)|\, dt$.

Exercise. Prove the arc length doesn't depend on parameterization.

Definition. If C is a differentiable arc parameterized by $z(t)$ with $z'(t) \neq 0$ on (a,b) define the unit tangent vector of C at $z(t)$ to be $\hat{T} = \dfrac{z'(t)}{|z'(t)|}$

Definition. An arc is smooth if $z'(t)$ is a continuous function.

Definition. A contour is an arc consisting of a finite number of smooth arcs joined end to end.

Theorem (Jordan Curve Theorem). Every simple closed contour divides the plane into two distinct domains, one of which is bounded (called the interior) while the other is unbounded (called the exterior)

Now we can define contour integrals of complex functions

Definition. If C is a contour represented by $z(t)$, $a \le t \le b$, and f is a complex function such that $f[z(t)]$ is piecewise continuous on $[a,b]$, then define the contour integral of f along C in terms of parameter t to be:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\cdot z'(t)\,dt$$

It can be varified by the change of variable formula that $\int_C f(z)\,dz$ is independent of the parameterization of C.

Remark. We can form a Riemann Sum Type definition of $\int_C f(z)\,dz$, which will be more intrinsic conceptually. Our definition is more practical here, since it tells us directly how to make the computation.

Definition. • If C is a contour $z(t)$, $a \le t \le b$, then $-C$ is its opposite contour. $z(-t)$, $-b \le t \le -a$, i.e. reverse the order of the points on C.

• If C_1 is a contour from z_1 to z_2, C_2 is a contour from z_2 to z_3, then $C_1 + C_2$ is the contour first going along C_1 and then going along C_2. If C_1 and C_2 have the same final point, we can define $C_1 - C_2 = C_1 + (-C_2)$

Proposition. (i) $\int_{-C} f\,dz = -\int_C f\,dz$

(ii) $\int_{C_1+C_2} f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$

(iii) $\int_{C_1-C_2} f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz$

(iv) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$

(v) $\int_C f(z)+g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

Proof. (i)

$$\int_{-C} f\,dz = \int_{-b}^{-a} f(z(-t))(-z'(-t))\cdot dt$$

change of variable $t=-\tau$

$$= -\int_b^a f(z(\tau))\cdot(-z'(\tau))\,d\tau$$

$$= -\int_a^b f(z(\tau))\cdot z'(\tau)\,d\tau$$

$$= -\int_C f\,dz$$

(ii) If C_1 is parameterized by $z_1(t)$, $a\le t\le b$, we can find a parameterization of C_2 such that $z_2(t)$, $b\le t\le c$.

So
$$\int_{C_1+C_2} f(z)\,dz = \int_a^c f(z(t))z'(t)\,dt = \int_a^b f(z_1(t))z_1'(t)\,dt + \int_b^c f(z_2(t))z_2'(t)\,dt$$
$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

The proofs for (iii), (iv), (v) are left as exercises

Example Let C be the unit circle path $z(t)=e^{it}$, $0\le t\le 2\pi$

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\cdot(e^{it})'\,dt = \int_0^{2\pi} i\cdot dt = 2\pi i$$

Example. C is a contour consisting of finitely many arcs in the order of $C_1, C_2, \ldots, C_n$, with a set of $n+1$ ordered endpoints $z_0, z_1, z_2, \ldots, z_n$ such that C_k begins at z_{k-1} and ends at z_k. Consider the integral

$$\int_C z\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz + \cdots + \int_{C_n} z\,dz$$

For C_k parameterized by $z(t)$, $a \le t \le b$ (so $z(a) = z_{k-1}$, $z(b) = z_k$)

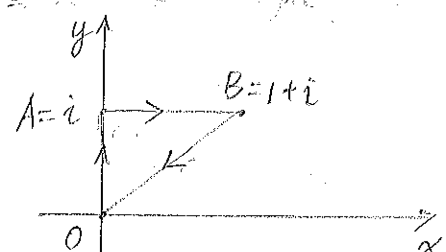
$$\int_{C_k} z\,dz = \int_a^b z(t)\cdot z'(t)\,dt$$

Note that $([z(t)]^2)' = 2z(t)\,z'(t)$ so $\left(\frac{z(t)^2}{2}\right)' = z(t)z'(t)$ by the generalized Fundamental Theorem of Calculus.

$$\int_{C_k} z\,dz = \int_a^b z(t)\,z'(t)\,dt = \frac{z(b)^2}{2} - \frac{z(a)^2}{2} = \frac{z_k^2}{2} - \frac{z_{k-1}^2}{2}$$

So $\displaystyle\int_C z\,dz = \sum_{k=1}^{n} \frac{z_k^2}{2} - \frac{z_{k-1}^2}{2} = \frac{z_n^2}{2} - \frac{z_0^2}{2}$

Example. Let C be the path along the triangle OAB in clockwise direction, starting and ending at O.



Then for a given function $f(z) = (y-x) - 3x^2 i$. $(z = x+iy)$

$$\int_C f(z)\,dz = \int_{\overline{OA}} f(z)\,dz + \int_{\overline{AB}} f(z)\,dz - \int_{\overline{OB}} f(z)\,dz$$

$\overline{OA}$ can be parameterized by $z_1(y) = yi$, $0 \le y \le 1$.

so $\displaystyle\int_{\overline{OA}} f(z)\,dz = \int_0^1 f(yi)\cdot z_1'(y)\,dy = \int_0^1 y\cdot i\,dy = \frac{1}{2}i$

$\overline{AB}$ can be parameterized by $z_2(x) = x + i$, $0 \le x \le 1$

$$\int_{\overline{AB}} f(z)\,dz = \int_0^1 f(x+i)\, z_2'(x)\,dx = \int_0^1 [(1-x) - 3x^2 i]\cdot 1\, dx$$
$$= \frac{1}{2} - i$$

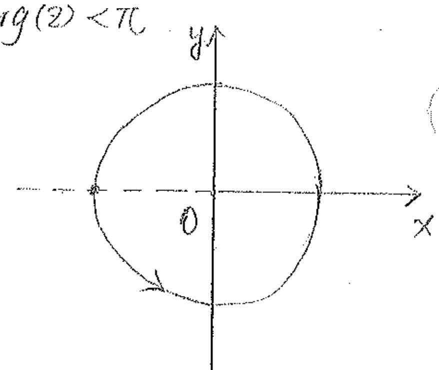
$\overline{OB}$ can be parameterized by $z_3(t) = t + ti$, $0 \le t \le 1$

$$\int_{\overline{OB}} f(z)\,dz = \int_0^1 f(t+ti)\, z'(t)\,dt = \int_0^1 -3t^2 i \cdot (1+i)\,dt = 1 - i$$

So $$\int_C f(z)\,dz = \int_{\overline{OA}} f(z)\,dz + \int_{\overline{AB}} f(z)\,dz - \int_{\overline{OB}} f(z)\,dz = \frac{i}{2} + (\frac{1}{2} - i) - (1 - i) = -\frac{1}{2} + \frac{1}{2}i$$

Example. Let C be the circle path $z(t) = e^{it}$, $-\pi \le t \le \pi$.

$f(z) = \log(z)$ is the branch $-\pi < \arg(z) < \pi$



Though the branch is not defined on the branch cut, removing one point from the integral doesn't affect the integral, so we get

$$\int_C f(z)\,dz = \lim_{\varepsilon \to 0} \int_{-\pi+\varepsilon}^{\pi-\varepsilon} \log(e^{it}) \cdot (e^{it})'\,dt$$

$$= \lim_{\varepsilon \to 0} \int_{-\pi+\varepsilon}^{\pi-\varepsilon} (it) \cdot i e^{it}\,dt$$

$$= \lim_{\varepsilon \to 0} \int_{-\pi+\varepsilon}^{\pi-\varepsilon} -t\cos t - it\sin t\,dt$$

$$= \lim_{\varepsilon \to 0} (-t\sin t - \cos t) - i(\sin t - t\cos t) \Big|_{-\pi+\varepsilon}^{\pi-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} -(\cos t + i\sin t) + i \cdot t(\cos t + i\sin t) \Big|_{-\pi+\varepsilon}^{\pi-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} -e^{it} + it\,e^{it} \Big|_{-\pi+\varepsilon}^{\pi-\varepsilon}$$

$$= -2\pi i.$$

UPPER BOUND

Lemma. If $w(t)$ is a piecewise continuous complex valued function $w(t): [a,b] \longrightarrow \mathbb{C}$, then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$$

Proof. If $\int_a^b w(t)\,dt = 0$, trivial.

If $\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \neq 0$ we see that

$r_0 = \left|\int_a^b w(t)\,dt\right|$. So the goal is to show

$r_0 \le \int_a^b |w(t)|\,dt$.

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \Rightarrow r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$$

$$r_0 \in \mathbb{R} \Rightarrow r_0 = \int_a^b Re[e^{-i\theta_0} w(t)]\,dt$$

$$\le \int_a^b |e^{-i\theta_0} w(t)|\,dt$$

$$= \int_a^b |w(t)|\,dt$$

Theorem. C is a contour. The arclength of C is $L \in \mathbb{R}^{\ge 0}$. $f(z)$ is a piecewise continuous function on C. If $M \ge 0$ such that $|f(z)| \le M$ for all points on C at which $f(z)$ is defined, then:

$$\left|\int_C f(z)\,dz\right| \le M \cdot L$$

Proof. For a piece of arc $C_i: z(t)$, $a \le t \le b$ in C.

By the Lemma, $\left|\int_{C_i} f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right| \le \int_a^b |f(z(t))| \cdot |z'(t)|\,dt$

Since $|f(z)| \le M$, we get

$$\int_a^b |f(z(t))| \cdot |z'(t)| dt \le \int_a^b M |z'(t)| dt = M \int_a^b |z'(t)| dt$$

$$= M \cdot L_i$$

A contour C is a union of several pieces of arcs $C_1, C_2, \ldots, C_n$

So $$\int_C f(z) dz = \sum_{i=1}^n \int_{C_i} f(z) dz \le \sum_{i=1}^n M L_i = M \sum_{i=1}^n L_i = ML.$$

Example. C_R is the semicircle $z = Re^{i\theta}$, $0 \le \theta \le \pi$

We will show $\lim_{R \to \infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz = 0$.

Note for R big enough, by triangle inequalities

$$|z+1| \le |z| + 1 = R + 1$$
$$|z^2+4| \ge |z|^2 - 4 = R^2 - 4$$
$$|z^2+9| \ge |z|^2 - 9 = R^2 - 9$$

So on C_R, $\left| \frac{z+1}{(z^2+4)(z^2+9)} \right| \le \frac{R+1}{(R^2-4)(R^2-9)}$

By the Theorem.

$$0 \le \left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz \right| \le \frac{R+1}{(R^2-4)(R^2-9)} \cdot \pi R = \pi \cdot \frac{R^2+R}{(R^2-4)(R^2-9)} = \pi \cdot \frac{1+\frac{1}{R}}{(1-\frac{4}{R^2})(R^2-9)}$$

$$0 \le \lim_{R \to \infty} \left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz \right| \le \lim_{R \to \infty} \pi \frac{1+\frac{1}{R}}{(1-\frac{4}{R^2})(R^2-9)} = 0$$

We conclude $\lim_{R \to \infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz = 0$

ANTIDERIVATIVE

Definition. $f(z)$ is a complex function. Define the antiderivative of $f(z)$ to be $F(z)$ if $F'(z) = f(z)$.

Example. If $f(z) = z^2$, then $F(z) = \frac{1}{3}z^3$ is an antiderivative of $f(z)$

Proposition If $f(z)$ is defined on a domain D, then the antiderivative of f on D is unique up to adding a constant.

Proof. If $F(z)$, $G(z)$ are both antiderivatives of $f(z)$ then $F'(z) = f(z)$ and $G'(z) = f(z)$

So $(F(z) - G(z))' = f(z) - f(z) = 0$ on D.

We know this implies $F(z) - G(z) \equiv C \in \mathbb{C}$ on D.

It's an interesting question to ask what functions have antiderivative. It turns out we can get equivalent contiditions by studying contour integrals.

Theorem. $f(z)$ is a continuous function on a domain D. The following are equivalent:

(a). $f(z)$ has antiderivative $F(z)$ on D.

(b). If C_1 and C_2 are two contours in D with same starting point and same terminal point, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

(c). If C is a closed contour in D, then $\int_C f(z)\,dz = 0$.

Proof. $(a) \Rightarrow (b)$

If $F'(z) = f(z)$, C is a contour in D from $z_1 \in D$ to $z_2 \in D$. We can parameterize C by some $z(t)$, $a \le t \le b$ (so $z(a) = z_1$, $z(b) = z_2$)

Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt = F(z(b)) - F(z(a))$$
$$= F(z_2) - F(z_1)$$

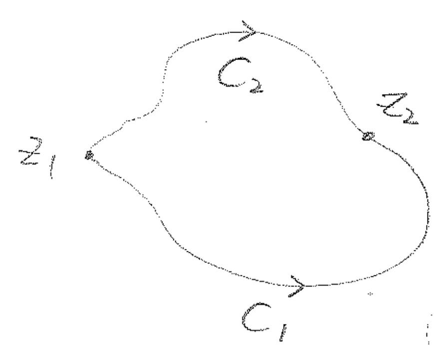
So we see $\int_C f(z)\,dz$ only depends on $F(z_2)$ and $F(z_1)$ i.e. the endpoints. If C_1 and C_2 both start at z_1 and terminates at z_2.

$$\int_{C_1} f(z)\,dz = F(z_2) - F(z_1) = \int_{C_2} f(z)\,dz$$

$(b) \Rightarrow (c)$

If C is a closed curve in D described by $z(t)$, $a \le t \le b$. Let $z_1 = z(a) = z(b)$ and $z_2 = z(c)$ for some $c \in (a, b)$. Let C_1 be the part of C from z_1 to z_2, i.e. C_1 is $z(t)$, $a \le t \le c$ and C_2 be the reversed path of z_2 to z_1 along C, i.e. C_2 is $z(-t)$, $-c \le t \le -b$.

Then $C = C_1 - C_2$ and C_1, C_2 both start at z_1 and terminates at z_2. So by assumption



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

(c) $\Rightarrow$ (b). If $\int_C f(z)\,dz = 0$ for any closed path C in D

then for any C_1, C_2 both start at z_1 and terminate at z_2. $C_1 - C_2$ is a closed path in C, so

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz = 0.$$

we get $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

(b) $\Rightarrow$ (a):

Fix a point $z_0 \in D$. define

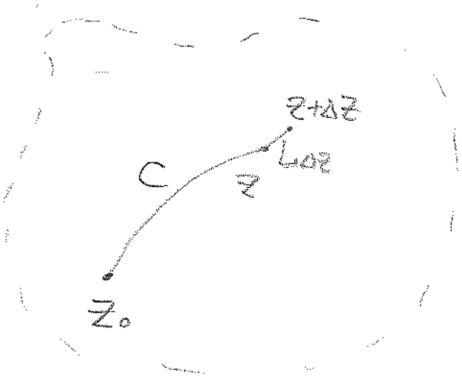
$$F(z) = \int_C f(z)\,dz$$ wher C is any contour from z_0 to z.

It's well-defined by the assumtion that all contour integrals only depend on the endpoints.

We need to verify $F'(z) = f(z)$

By definition, $F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$

We take $|\Delta z|$ to be small so that the line segment $L_{\Delta z}$ from z to $z + \Delta z$ is in D.



Then $F(z+\Delta z) - F(z)$

$$= \int_{C + L_{\Delta z}} f(z)\,dz - \int_C f(z)\,dz$$

$$= \int_{L_{\Delta z}} f(z)\,dz$$

$$= \int_0^1 f(z + t\Delta z) \cdot \Delta z\,dt$$ (Note $L_{\Delta z}$ can be expressed as $z(t) = z + t\Delta z$, $0 \le t \le 1$)

$$= \Delta z \int_0^1 f(z + t\Delta z)\,dt$$

This implies $\dfrac{F(z+\Delta z)-F(z)}{\Delta z} = \int_0^1 f(z+t\Delta z)\,dt$

We have assumed $f(z)$ is a continuous function, so for any $\varepsilon>0$, $\exists\ \delta>0$ such that $|\Delta z|<\delta \Rightarrow |f(z+\Delta z)-f(z)|<\varepsilon$

Then $\left| \dfrac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) \right|$

$$= \left| \int_0^1 f(z+t\Delta z)\,dt - \int_0^1 f(z)\,dt \right|$$

$$\le \int_0^1 |f(z+t\Delta z)-f(z)|\,dt$$

$$< \int_0^1 \varepsilon\,dt$$

$$= \varepsilon \quad \text{for any } 0<|\Delta z|<\delta.$$

So we conclude $\lim\limits_{\Delta z\to 0} \dfrac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$

i.e. $F'(z)=f(z)$.

Corollary. If $F(z)$ is the antiderivative of $f(z)$, then for any contour C from z_1 to z_2 in a domain D,

$$\int_C f(z)\,dz = F(z_2)-F(z_1)$$

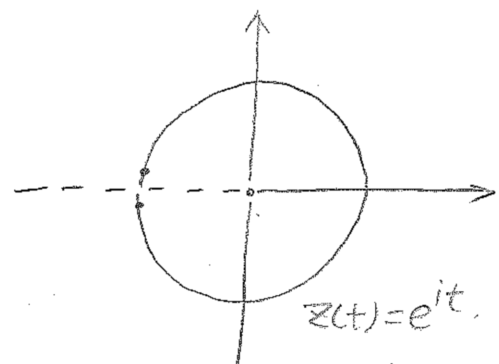
We've proved it during the proof of the theorem (Part (a) $\Rightarrow$ (b))

Example. If C is the circle $z(t)=e^{it}$, $0\le t\le 2\pi$, then $\int_C \frac{1}{z^2}\,dz=0$ since $\frac{1}{z^2}$ has antiderivative $-\frac{1}{z}$ on the domain $\mathbb{C}\setminus\{0\}$.

Note: But $\int_C \frac{1}{z}dz \neq 0$ since $\frac{1}{z}$ doesn't have antiderivative on $\mathbb{C}\setminus\{0\}$: Recall that $(\log z)' = \frac{1}{z}$ for any branch of $\log z$, but we need to make a branch cut to get a branch, so $\log z$ is not a single-valued analytic function on $\mathbb{C}\setminus\{0\}$. it cannot be used as the antiderivative of $\frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$.

However, we can use the limit trick to apply the antiderivative method:

We take the branch of $\log z$: $-\pi < \theta < \pi$, then $\log z$ is the antiderivative of $\frac{1}{z}$ on this domain $\mathbb{C}\setminus\mathbb{R}^{\leq 0}$.



$$\int_C \frac{1}{z}dz = \lim_{\varepsilon\to 0}\int_{z(-\pi+\varepsilon)}^{z(\pi-\varepsilon)} \frac{1}{z}\,dz$$

$$= \lim_{\varepsilon\to 0} \log(z(\pi-\varepsilon)) - \log(z(-\pi+\varepsilon))$$

$$= \lim_{\varepsilon\to 0} i(\pi-\varepsilon) - i(-\pi+\varepsilon)$$

$$= 2\pi i$$

CAUCHY - GOURSAT THEOREM

Theorem. (Early Version by Cauchy)
C is a simple closed contour, R is the set of all points interior to or on C. If f is analytic on R, and f' is continuous, then $\int_C f(z)\,dz = 0$.

Proof. We let $f(z) = u(x,y) + iv(x,y)$ and C be $z(t) = x(t) + iy(t)$
$a \le t \le b$.

Then

$$\int_C f(z)\,dz = \int_a^b \big(u(x(t),y(t)) + i\,v(x(t),y(t))\big).\big(x'(t) + iy'(t)\big)\,dt$$

$$= \int_a^b u(x(t),y(t))\,x'(t) - v(x(t),y(t))\cdot y'(t)\ \ dt\ +$$

$$i\int_a^b u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\ dt$$

$$= \int_C u\,dx - v\,dy \ +\ i\int_C v\,dx + u\,dy$$

Green's Theorem

$$\stackrel{\downarrow}{=} \left(\iint_R \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\ dA\right) + i\left(\iint_R \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,dA\right)$$

$$= \iint_R -(v_x + u_y)\,dA + i\iint_R (u_x - v_y)\,dA$$

Cauchy-Riemann Equations

$$\stackrel{\downarrow}{=} 0$$

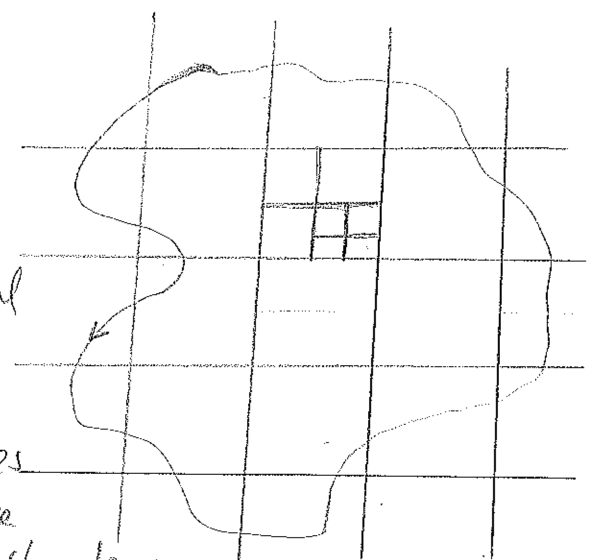
Remark. In this early version, f' is assumed to be continuous because we need to satisfy the condition for Green's Theorem

Lemma. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C, together with the points on C itself. For any positive number $\varepsilon > 0$, the region R can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \dots, n$ such that in each one there is a fixed point z_j for which the inequality

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon$$

is satisfied by all points other than z_j in that square or partial square.

Proof. We first construct equally spaced horizontal and vertical lines that separate the region R into squares and partial squares.



If in one of the squares or partial squres, we cannot find a z_j, then we subdivide it into four smaller squares of equal size.

We claim that after finitely times of subdivisions, each small square will satisfy the requirements of the lemma.

We'll prove by contradiction. Suppose σ_k, $k \in \mathbb{N}$ is a nested sequence of squares, the diameter of σ_k is $\frac{b}{2^k}$, where b is a constant. It can be shown that there exits a $z_0 \in \bigcap_{k=1}^{\infty} \sigma_k$.

Since f is analytic at z_0, for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$.

but then we see if K is big enough, σ_K has diameter less than ε. then for any $z \in \sigma_K$, $\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$.

Contradict to the assumption.

Theorem (Cauchy - Goursat)

If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

Proof By the previous Lemma, given any $\varepsilon > 0$. we can cover the region R enclosed by C by squares or partial squares σ_j such that $\exists\ z_j \in \sigma_j$. $\forall\ z \in \sigma_j \setminus \{z_j\}$

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon .$$

Define

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & \text{if } z \in \sigma_j \setminus \{z_j\} \\ 0 & \text{if } z = z_j \end{cases}$$

Then δ_j is a continuous function on σ_j. such that $|\delta_j(z)| < \varepsilon$

We can rewrite the above as

$$f(z) = f(z_j) + (\delta_j(z) + f'(z_j))(z - z_j)$$

$$= f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z)$$

We thus get: $f(z)$ integrates along C_j, the boundary of σ_j (conterclockwise)

$$\int_{C_j} f(z)\,dz = \int_{C_j} f(z_j) - z_j f'(z_j) + f'(z_j)z + (z-z_j)\delta_j(z)\,dz$$

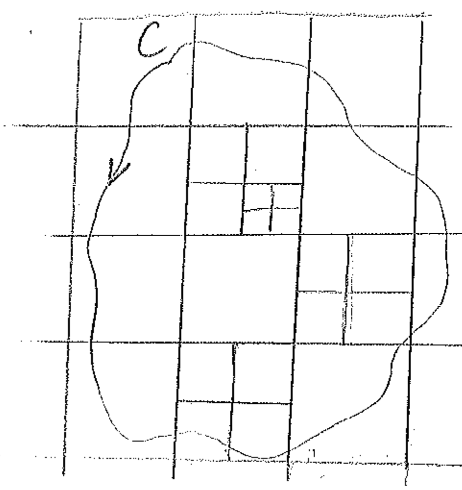
$$= \left(f(z_j) - z_j f'(z_j)\right)\int_{C_j} 1\,dz + f'(z_j)\int_{C_j} z\,dz$$

$$+ \int_{C_j} (z-z_j)\delta_j(z)\,dz$$

$$= \int_{C_j} (z-z_j)\delta_j(z)\,dz$$

Another key observation is that

$$\int_C f(z)\,dz = \sum_j \int_{C_j} f(z)\,dz = \sum_j \int_{C_j} (z-z_j)\delta_j(z)\,dz$$



If C_j is a square, let s_j be the length of an edge of σ_j. A_j is the area of σ_j, then

$$\left|\int_{C_j} (z-z_j)\delta_j(z)\,dz\right| \le \int_{C_j} |z-z_j|\,|\delta_j(z)|\,dz$$

$$\le (\sqrt{2}\, s_j\, \varepsilon)\cdot(4 s_j)$$

$$= 4\sqrt{2}\,\varepsilon A_j$$

If C_j is the boundary of a partial square.

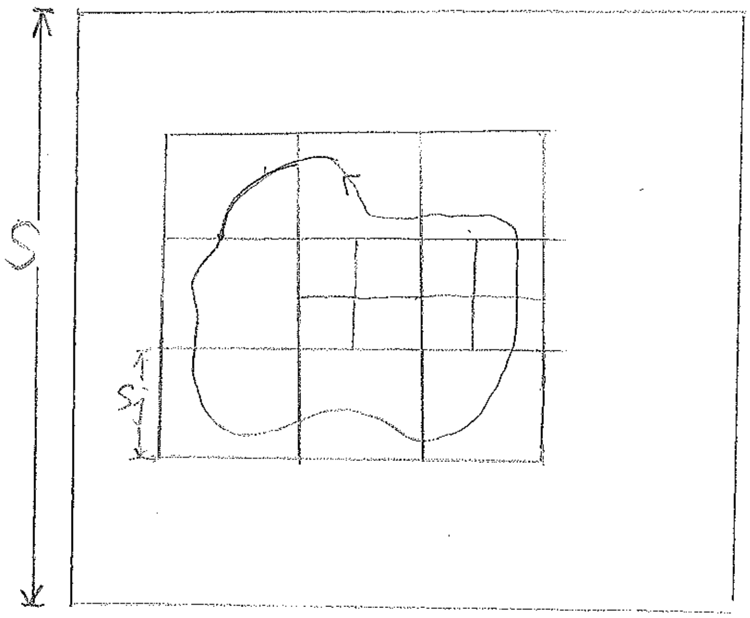
$$\left|\int_{C_j} (z-z_j)\delta_j(z)\,dz\right| \le (\sqrt{2}\, S_j\, \varepsilon)\cdot(4S_j + L_j)$$

$$< 4\sqrt{2}\, A_j\, \varepsilon + \sqrt{2}\, S L_j\, \varepsilon$$

where L_j is the part of C_j which is also part of C.

S is the length of a side of some square that encloses the entire contour C as well as all of the squares covering R. Now let

L be the arc length of C



$$\left|\int_C f(z)\,dz\right| \le \sum_j \left|\int_{C_j} f(z)\,dz\right|$$

$$\le 4\sqrt{2}\,\varepsilon S^2 + \sqrt{2}\,S L \varepsilon$$

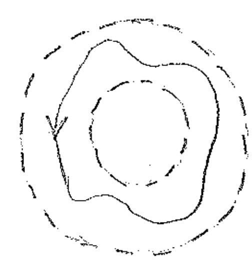
$$= (4\sqrt{2}\,S^2 + \sqrt{2}\,SL)\,\varepsilon$$

Since S and L are constants, we see as $\varepsilon \to 0$, we get

$$\int_C f(z)\,dz = 0$$

Definition. A simply connected domain D is a domain such that every simple closed contour within it encloses only points of D.

Example.
- $\mathbb{C}$ is a simply connected domain
- The interior of a simple closed curve is a simply-connected domain
- The annulus region $1 < |z| < 2$ is NOT a simply-connected domain



Theorem. If f is analytic throughout a simply-connected domain D, then $\int_C f(z)\,dz = 0$ for every closed contour C lying in D.

Proof. If C is simple closed in the simply-connected domain D, then C and its interior are all in D, so f is analytic on C and in its interior, by Cauchy-Goursat Theorem $\therefore \int_C f(z)\,dz = 0$

More generally, if C has finitely many self-intersections, we can regard C as a sum of simple closed contours. The result also follows.

If there're infinitely many self-intersection, we need more tricks, we'll not discuss here.

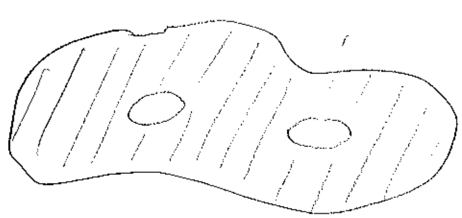
Example. $\int_C \frac{1}{z}\,dz = 0$ for all simple closed contour C that doesn't enclose the origin.

Corollary. If f is an analytic function throughout a simply-connected domain D, then f has an antiderivative on D

Corollary. Entire functions have antiderivatives

Definition. If a domain is not simply-connected, we say it's multiply connected

Example.



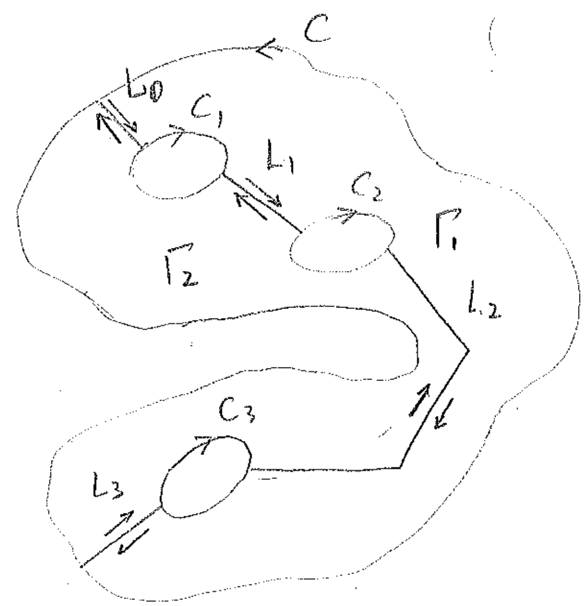
Theorem. If C is a simple closed contour, counterclockwisely oriented, and C_k $(k=1,2,\dots,n)$ are simple closed contours interior to C, all in the clockwise direction, that are disjoint and whose interior have no points in common.

If f is analytic in the multiply connected domain consisting of points inside C and exterior to each C_i, then:

$$\int_C f(z)\,dz + \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0$$

Proof. We use polygonal paths to connect C to C_1, C_1 to C_2, ..., C_{n-1} to C_n, and C_n to C, and label these paths by $L_0, L_1, \dots, L_n$.

Then two simple closed paths Γ_1 and Γ_2 are formed.



By Cauchy-Goursat,

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

The integral on L_i cancel with each other, so we get

$$\int_C f(z)\,dz + \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0$$

Corollary. (Principle of Deformation of Paths)

Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If f is analytic in the closed region consisting of those contours and all points between them, then:

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Remark. More generally, in the theorem, if we take $C_1, \dots, C_n$ counterclockwisely oriented, and all the other conditions unchanged, then:

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

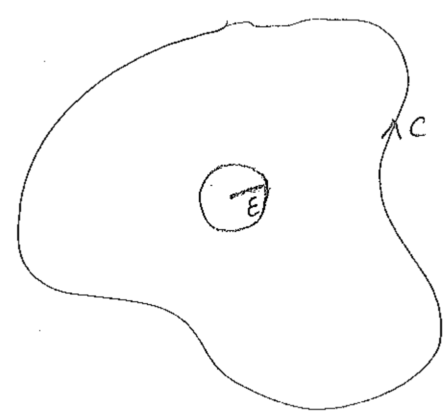
Example. If C is a simple closed contour counterclockwisely oriented, enclosing 0 in its interior, then

claim: $\displaystyle\int_C \frac{1}{z}\,dz = 2\pi i$.

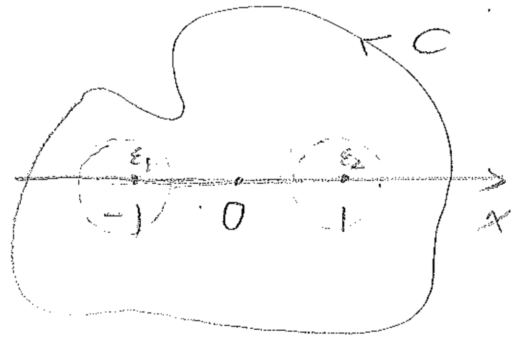
We can take a circle C_ε of small radius $\varepsilon > 0$ centered at 0 such that the circle is in the interior of C.

Then By the Corollary.

$$\int_C \frac{1}{z}\,dz = \int_{C_\varepsilon} \frac{1}{z}\,dz = 2\pi i$$



Example. C is a simple closed contour, counterclockwisely oriented, enclosing 1 and -1 in its interior. Let's try to compute $\int_C \frac{1}{z^2-1}\,dz$.



Take small $\varepsilon_1>0$ and $\varepsilon_2>0$ such that the circle

C_1: $z_1(t) = -1+\varepsilon_1 e^{it}$, $0\le t\le 2\pi$ and

C_2: $z_2(t) = 1+\varepsilon_2 e^{it}$, $0\le t\le 2\pi$ are

in the interior of C and C_1, C_2 disjoint.

Then since $\frac{1}{z^2-1}$ is analytic inside C and exterior of C_1, C_2

$$\int_C \frac{1}{z^2-1}\,dz = \int_{C_1}\frac{1}{z^2-1}\,dz + \int_{C_2}\frac{1}{z^2-1}\,dz$$

$$= \frac{1}{2}\left[\int_{C_1}\frac{1}{z-1}\,dz - \int_{C_1}\frac{1}{z+1}\right]$$

$$+\frac{1}{2}\left[\int_{C_2}\frac{1}{z-1}\,dz - \int_{C_2}\frac{1}{z+1}\,dz\right]$$

$$= -\frac{1}{2}\int_{C_1}\frac{1}{z+1}\,dz + \frac{1}{2}\int_{C_2}\frac{1}{z-1}\,dz$$

$$= -\frac{1}{2}\cdot\int_0^{2\pi}\frac{1}{\varepsilon_1 e^{it}}\cdot i\varepsilon_1 e^{it}\,dt + \frac{1}{2}\int_0^{2\pi}\frac{1}{\varepsilon_2 e^{it}}\,i\varepsilon_2 e^{it}\,dt$$

$$= -\pi i + \pi i$$

$$= 0$$

CAUCHY INTEGRAL FORMULA

The Cauchy Integral Formula is a fundamental result in Complex analysis, which tells us how the value of an analytic function at $z_0 \in \mathbb{C}$ is related to integral around it.

Theorem. (Cauchy Integral Formula)
Let f be analytic everywhere inside and on a simple closed Contour C, taken counterclockwise direction. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Proof. Take a circle C_r of small radius $r>0$, centered at z_0, counterclockwise oriented and C_r is in the interior of C.

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = \int_{C_r} \frac{f(z)}{z-z_0}\,dz - \int_{C_r}\frac{f(z_0)}{z-z_0}\,dz$$

$$= \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

f is analytic at z_0, so it's continuous at z_0. For any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$.

Now take $\delta' = \min\{r, \delta\}$, then

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| = \left|\int_{C_{\delta'}} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\delta'}\cdot 2\pi\delta' = 2\pi\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = 0,$$

i.e. $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Example. C is the positively oriented circle $|z|=1$.

$$\int_C \frac{\cos z}{z(z^2+9)}\,dz = \int_C \frac{\frac{\cos z}{z^2+9}}{z-0}\,dz = 2\pi i \cdot \frac{\cos 0}{0^2+9} = \frac{2\pi i}{9}$$

Theorem. f is analytic inside and on a simple closed contour C, which is positively oriented. If z_0 is in the interior of C, then for each $n \in \mathbb{N}$, $f^{(n)}(z_0)$ exists and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Proof. We will show the case for $n=1$. The general case can be proved by induction using similar idea, but more technical details will be involved. The reader may refer to Ahlfors' Complex Analysis book for more details.

We know $f'(z_0) = \lim\limits_{\Delta z\to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z\to 0} \frac{1}{2\pi i\Delta z}\left[\int_C \frac{f(z)}{z-z_0-\Delta z}\,dz - \int_C \frac{f(z)}{z-z_0}\,dz\right]$$

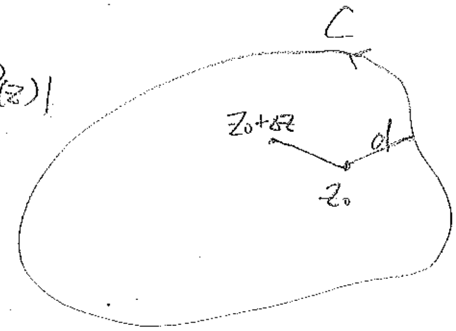
$$= \lim_{\Delta z\to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{\Delta z}\cdot\left(\frac{1}{z-z_0-\Delta z} - \frac{1}{z-z_0}\right)dz$$

$$= \lim_{\Delta z\to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz$$

$$f'(z_0) - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz = \lim_{\Delta z\to 0}\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} - \frac{f(z)}{(z-z_0)^2}\,dz$$

$$= \lim_{\Delta z\to 0}\frac{1}{2\pi i}\int_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz$$

Let $d = \min_{z \in C} |z - z_0|$ and L is the arclength of the contour C, $M = \max_{z \in C} |f(z)|$.



We may only consider the case $|\Delta z| < d$, since we'll take the limit as $\Delta z \to 0$.

so $\displaystyle \left| \int_C \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2} dz \right| \leq \frac{|\Delta z| \cdot M}{(d - |\Delta z|) d^2} \cdot L$,

(since $|z - z_0 - \Delta z| \geq \big| |z - z_0| - |\Delta z| \big| \geq d - |\Delta z| > 0$)

We now see as $\Delta z \to 0$, $\displaystyle \int_C \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2} dz \to 0$

so $\displaystyle f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz$

Example. Compute $\displaystyle \int_C \frac{e^z}{z^4} dz$, where C is the unit circle $|z| = 1$ positively oriented.

$$\int_C \frac{e^z}{z^4} dz = \frac{2\pi i}{3!} \cdot \frac{3!}{2\pi i} \int_C \frac{e^z}{(z-0)^{3+1}} dz = \frac{2\pi i}{3!} \cdot \frac{d^3 e^z}{dz^3}\bigg|_{z=0} = \frac{\pi}{3} i$$

Remark. We sometimes want to express the function $f^{(n)}(z)$ by the Cauchy Integral Formula, so we need to change the variables in the previous case:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s - z)^{n+1}} ds$$

Theorem. If f is analytic at z_0, then f' is analytic at z_0.

Proof. If f is analytic at z_0, by definition, there's a open neighbourhood U of z_0 on which f is analytic. So we can take a small circle C centred at z_0 such that $C \subseteq U$. Then by Cauchy Integral Formula, for each z inside the circle, $f''(z)$ exists and

$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds$$

so f' is analytic at z_0.

Corollary. If f is analytic at z_0, then for each $n \in \mathbb{N}$, $f^{(n)}$ is analytic at z_0.

Corollary. If $f(z) = u(x,y) + i v(x,y)$ is analytic at $z_0 = (x_0, y_0)$, then all the partial derivatives of any order of u and v exist at (x_0, y_0).

Theorem. f is continuous on a domain D. If $\int_C f(z)\,dz = 0$ for every closed contour inside D, then f is analytic on D.

Proof. $\int_C f(z)\,dz = 0$ for every closed curve on $D \Rightarrow f$ has an antiderivative F on D, i.e. $F'(z) = f(z)$.

Since f is the derivative of F, $f = F'$ is analytic at every point in D, so f is analytic on D.

Theorem. (Cauchy's Inequality) f is analytic inside and on a positively oriented circle C_R centred at z_0 with radius R. $M_R = \max_{z \in C} |f(z)|$, then: $|f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n}$

Proof. By Cauchy Integral Formula,

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R$$

$$= \frac{n!\, M_R}{R^n}$$

Theorem. (Liouville's Theorem) If f is an entire function that is bounded, then $f(z) \equiv c$ for some $c \in \mathbb{C}$

Proof. Since f is bounded, there's $M > 0$ such that $|f(z)| < M$ for all $z \in \mathbb{C}$

f is entire, so for any $z \in \mathbb{C}$, we take the circle C_R centred at z with radius R, and apply the previous Theorem:

$$|f'(z)| \leq \frac{M_R}{R} < \frac{M}{R}$$

Since this is true for any $R > 0$, letting $R \to +\infty$ we get $|f'(z)| = 0$. i.e $f'(z) = 0 \ \forall z \in \mathbb{C}$

We conclude f is a constant function

Theorem. (Fundamental Theorem of Algebra)
Any non constant polynomial $p(z) \in \mathbb{C}[z]$ has at least one root in $\mathbb{C}$. i.e $\exists z_0 \in \mathbb{C}$ such that $p(z_0) = 0$

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a nonconstant polynomial. Suppose $p(z)$ has no root, then $\frac{1}{p(z)}$ is an entire function.

If we can show $\frac{1}{p(z)}$ bounded, then by Liouville's Theorem, $\frac{1}{p(z)} \equiv C \Rightarrow p(z) \equiv \frac{1}{C}$. Contradicts to the assumption, we'll finish the proof.

let $w = \frac{p(z)}{z^n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}$

so $|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|}$

We can take a large $R > 0$ such that whenever $|z| \geq R$,

$$\frac{|a_0|}{|z|^n} < \frac{|a_n|}{2n}, \quad \frac{|a_1|}{|z|^{n-1}} < \frac{|a_n|}{2n}, \quad \ldots, \quad \frac{|a_{n-1}|}{|z|} < \frac{|a_n|}{2n}$$

$$|w| \leq n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}$$

so for $|z| > R$, $|P(z)| = |a_n + w| \cdot |z^n| \geq \left(|a_n| - \frac{|a_n|}{2}\right) \cdot R^n$

$$= \frac{|a_n|}{2} \cdot R^n$$

we see $\left|\frac{1}{p(z)}\right|$ is bounded for $|z| > R$.

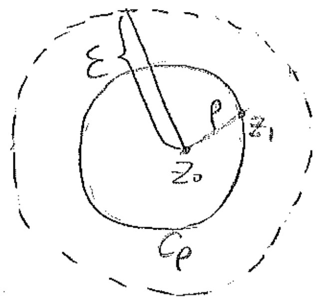
On $|z| \leq R$, $\left|\frac{1}{p(z)}\right|$ is also bounded since $|z| \leq R$ is a bounded and closed region and $\frac{1}{p(z)}$ is continuous.

So we conclude $\frac{1}{p(z)}$ is bounded

Lemma. Suppose that $|f(z)| \le |f(z_0)|$ at each point z in some neighbourhood of z_0: $|z-z_0|<\varepsilon$ in which f is analytic. Then $f(z) \equiv f(z_0)$ on this neighbourhood.

Proof For any z_1 in the neighbourhood $|z-z_0|<\varepsilon$.
let $\rho = |z_1 - z_0| < \varepsilon$, C_ρ is the circle $z(t) = z_0 + \rho e^{it}$, $0 \le t \le 2\pi$.

By assumption, f is analytic on and inside C_ρ, so by Cauchy Integral Formula,



$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{it})}{\rho e^{it}} \cdot i \cdot \rho e^{it}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it})\,dt$$

so $$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\rho e^{it})\,dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$$

$$\le \frac{1}{2\pi}\cdot|f(z_0)|\cdot 2\pi$$

$$= |f(z_0)|$$

we get $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| - |f(z_0+\rho e^{it})|\,dt = |f(z_0)| - \frac{1}{2\pi}\int_0^{2\pi}|f(z_0+\rho e^{it})|\,dt = 0.$$

but $|f(z_0)| - |f(z_0+\rho e^{it})| \ge 0$ for all $t \in [0, 2\pi]$
we conclude

$$|f(z_0)| = |f(z_0+\rho e^{it})| \quad \forall\ t \in [0, 2\pi]$$

Since the above holds for any $\rho < \varepsilon$, we conclude

$$|f(z_0)| = |f(z)| \text{ for any } |z - z_0| < \varepsilon.$$

$\Rightarrow$ $|f(z)|$ is constant on the domain $|z - z_0| < \varepsilon$.

Recall we've proved long ago in an example that this implies $f(z) \equiv C$ on the domain.

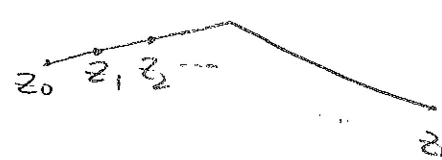
So we finish the proof.

Theorem. (Maximum Modulus Principle) If a function f is analytic and not constant in a domain D, then $|f(z)|$ doesn't have a maximum in D.

Proof. Suppose $f(z)$ takes maximum at $z_0 \in D$, i.e. $|f(z)| \leq |f(z_0)|$ for any $z \in D$.

Now for any $z \in D$, we can connect z_0 to z by a polygonal path L.

let d be the shortest distance between points on L and boundary of D



(In case $D = \mathbb{C}$, we can take any $d > 0$)

Then L is covered by finitely many open balls of radius d, with centre at $z_0, z_1, \ldots, z_n = z$, respectively, such that $|z_i - z_{i-1}| < d$ for all $i = 1, 2, \ldots, n$.

By the Lemma, f is constant on $B(z_0, d)$, and $z_1 \in B(z_0, d)$ so $f(z_1) = f(z_0) \geq f(z)$ for any $z \in D$. Apply the Lemma to $B(z_1, d)$

we get f is constant on $B_1(z_0, d)$, and we continue this process n times to conclude $f(z) = f(z_n) = f(z_{n-1}) = \cdots = f(z_0)$

Since $z \in D$ is arbitrary at beginning, we conclude f is constant

Corollary. If f is continuous on a bounded closed region R and f is analytic and not constant in the interior of R. Then $\max_{z\in R}|f(z)|$ is obtained only at some points on the boundary of R

Proof. Since f is continuous and R is closed and bounded, $\max_{z\in R}|f(z)|$ exists. If it appears in the interior, then by the Maximum Modulus Principle, f is constant in the interior of R, contradiction.

Remark. In the proof of the Lemma, the formula

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})\, dt$$

is called the Gauss' Mean Value Theorem. It tells us if f is analytic on and inside a circle $|z - z_0| = \rho$, then $f(z_0)$ is the arithmetic mean of the values on the circle.

SERIES

Definition. An infinite sequence $\{z_n\}$ of complex numbers has limit z if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |z_n - z| < \varepsilon.$$

We say $\{z_n\}$ converges to z if $\{z_n\}$ has limit z, and denoted by $\lim_{n\to\infty} z_n = z$.

We can study the convergence of a sequence by studying its real and imaginary parts.

Theorem. $z_n = x_n + iy_n$ is a sequence and $z = x + iy$ is a complex number. Then

$$\lim_{n\to\infty} z_n = z \quad \text{if and only if} \quad \lim_{n\to\infty} x_n = x \ \& \ \lim_{n\to\infty} y_n = y.$$

Proof "$\Leftarrow$" If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$,

then for any $\varepsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ & $N_2 \in \mathbb{N}$ such that $n > N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$,

$$n > N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2}$$

So for any $n > \max\{N_1, N_2\}$

$$|z_n - z| \le |x - x_n| + |y - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $\lim_{n\to\infty} z_n = z$

"$\Rightarrow$" If $\lim_{n\to\infty} z_n = z$

then for any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $n > N \Rightarrow |z_n - z| < \varepsilon$

For any $n > N$

$|x_n - x| \leq |z_n - z| < \varepsilon$ and $|y_n - y| \leq |z_n - z| < \varepsilon$

so $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Example. $z_n = -1 + i \cdot \frac{(-1)^n}{n^2}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-1) + i \lim_{n\to\infty} \frac{(-1)^n}{n^2} = -1 + i \cdot 0 = -1$$

You may also use definition to conclude $\lim_{n\to\infty} z_n = -1$:

$\forall\, \varepsilon > 0$, let $N > \frac{1}{\sqrt{\varepsilon}}$,

$$n > N \Rightarrow |z_n - (-1)| = \left| i \cdot \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2} < \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2} = \varepsilon$$

Remark. z_n converges to z doesn't necessarily imply $\mathrm{Arg}(z_n)$ converges to $\mathrm{Arg}(z)$.

We may take the above example again

It's easy to see

$$\lim_{k\to\infty} \mathrm{Arg}(z_{2k-1}) = -\pi$$

but $$\lim_{k\to\infty} \mathrm{Arg}(z_{2k}) = \pi$$

so $\mathrm{Arg}(z_n)$ is not convergent

Exercise. Prove that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} |z_n| = |z|$

Definition. An infinite series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$ of complex numbers converges to the sum S if the sequence of partial sums $S_N = \sum_{n=1}^{N} S_n$ converges to S.

We write $\sum_{i=1}^{\infty} z_n = S$

Theorem. $z_n = x_n + iy_n$ is a sequence and $S = X + iY$ is a complex number. Then

$$\sum_{n=1}^{\infty} z_n = S \text{ if and only if } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

Proof. Let $S_N = \sum_{n=1}^{N} z_n$, $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$

Then $S_N = X_N + iY_N$

So $\lim_{N\to\infty} S_n = S$ if and only if $\lim_{N\to\infty} X_N = X$ & $\lim_{N\to\infty} Y_N = Y$

i.e. $\sum_{i=1}^{\infty} z_n = S$ if and only if $\sum_{i=1}^{\infty} x_n = X$ & $\sum_{i=1}^{\infty} y_n = Y$

Corollary. If $\sum_{i=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Proof Write. $z_n = x_n + iy_n$. If $\sum_{i=1}^{\infty} z_n$ converges

$\sum_{i=1}^{\infty} z_n = \sum_{i=1}^{\infty} x_n + i \cdot \sum_{i=1}^{\infty} y_n$, $\sum_{i=1}^{\infty} x_n$ & $\sum_{i=1}^{\infty} y_n$ are convergent.

So $\lim_{n\to\infty} x_n = 0$ & $\lim_{n\to\infty} y_n = 0$

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0.$

Definition. A series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Proposition. An absolutely convergent series is convergent.

Proof. If $\sum_{n=1}^{\infty} |z_n|$ is convergent, then by the Comparison Test,

$\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ are convergent since

$|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$ for all n.

We see $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent since they're absolutely convergent, so we conclude

$\sum_{n=1}^{\infty} z_n$ is convergent

Definition. A power series is a series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n \in \mathbb{C}$, $z_0 \in \mathbb{C}$ and z is the variable.

Example $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for any $|z|<1$.

$S_N = \sum_{n=1}^{N} z^N$. define $\rho_N = \frac{1}{1-z} - \sum_{n=1}^{N} z^N$

To show $\lim_{N\to\infty} S_N = \frac{1}{1-z}$ is the same as to show

$\lim_{N\to\infty} \rho_N = 0$

$S_N = \sum_{n=1}^{N} z^N = \frac{1-z^{N+1}}{1-z}$

$$\rho_N(z) = \frac{1}{1-z} - \frac{1-z^{N+1}}{1-z} = \frac{z^{N+1}}{1-z}$$

we see for each fixed $|z|<1$,

$$|\rho_N(z)| = \frac{|z|^{N+1}}{|1-z|} \longrightarrow 0 \quad \text{as } N \longrightarrow \infty$$

so $\lim\limits_{N\to\infty} \rho_N(z) = 0$, $\lim\limits_{n\to\infty} S_N = \frac{1}{1-z}$. i.e $\sum\limits_{n=1}^{\infty} z^n = \frac{1}{1-z}$

Observation. In the example above, we proved

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } |z|<1.$$

Let $f(z) = \frac{1}{1-z}$ defined in the open disk $|z|<1$, $f(z)$ is analytic on this disk.

$$f'(z) = \frac{1}{(1-z)^2}, \quad f''(z) = \frac{2}{(1-z)^3}, \quad \cdots, \quad f^{(k)}(z) = \frac{k!}{(1-z)^{k+1}}, \cdots$$

So $f'(0) = 1$, $f''(0) = 2$, $\cdots$, $f^{(k)}(0) = k!$, $\cdots$

The equation tells us

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^2$$ on the disk centred at 0 with radius 1.

Definition. f is a function that is analytic at z_0, then define the Taylor Series of f at z_0 to be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

TAYLOR SERIES

In the previous example, we see for the analytic function $f(z)=\frac{1}{1-z}$ on the disk $|z|<1$, $f(z)$ equals its Talyor Series at $z_0=0$. Now we would like to see in general if there is such kind of results.

Theorem. f is analytic throughout a disk $|z-z_0|<R_0$ (centred at z_0 and with radius R_0). Then:

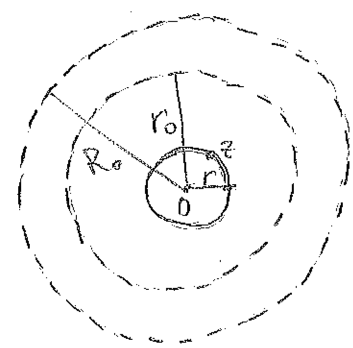
$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{for all } |z-z_0|<R_0$$

Proof. We will first show the case $z_0=0$, then obtain the general case.

For any $|z|<R_0$, denote $r=|z|$.

Pick $r<r_0<R_0$, and denote the circle $|z|=r_0$ to be C_0, positively oriented.

f is analytic on and inside C_0, so Cauchy Integral Formula:



$$f(z)=\frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{s-z}\,ds$$

$$=\frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{s}\cdot\frac{1}{1-\frac{z}{s}}\,ds$$

Recall that in the previous example, we have shown that for any $|z|<1$, $\frac{1}{1-z}=\sum_{n=0}^{N-1}z^n+\frac{z^N}{1-z}$

So $\frac{1}{1-\frac{z}{s}}=\sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n+\frac{\left(\frac{z}{s}\right)^N}{1-\left(\frac{z}{s}\right)}=\sum_{n=0}^{N-1}\frac{z^n}{s^n}+\frac{z^N s}{s^N(s-z)}$

So $f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s} \cdot \frac{1}{1-\frac{z}{s}} \, ds$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s}\left(\sum_{n=0}^{N-1} \frac{z^n}{s^n} + \frac{z^N s}{s^N(s-z)}\right) ds$$

$$= \sum_{n=0}^{N-1} \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s^N(s-z)} ds \cdot z^N$$

$$= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \frac{z^N}{2\pi i}\int_{C_0} \frac{f(s)}{s^N(s-z)} ds$$

We need to show $\lim_{N\to\infty} \frac{z^N}{2\pi i}\int_{C_0} \frac{f(s)}{s^N(s-z)} ds = 0$

Let $M = \max_{s\in C_0} f(s)$. For any $|s| = r_0$ and any $|z| = r$,

$|s-z| \geq \big||s| - |z|\big| = r_0 - r$.

So $\left|\frac{z^N}{2\pi i}\int_{C_0} \frac{f(s)}{s^N(s-z)} ds\right| \leq \frac{r^N}{2\pi} \cdot \frac{M}{r_0^N \cdot (r_0 - r)} \cdot 2\pi r_0 = \frac{M r_0}{r_0 - r} \cdot \left(\frac{r}{r_0}\right)^N \to 0$

as $N \to \infty$ we get $\lim_{N\to\infty} \frac{z^N}{2\pi i}\int_{C_0} \frac{f(s)}{s^N(s-z)} ds = 0$

So $f(z) = \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$.

Next, we prove the general case.

Let $g(z) = f(z+z_0)$, then $g(z)$ is analytic inside the circle $|z| = R_0$. By the $z_0 = 0$ case just proved,

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

$$\Rightarrow f(z+z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

Corollary. f is analytic at z_0, then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ in some neighbourhood of z_0

Corollary. f is an entire function. $z_0 \in \mathbb{C}$. then for any $z \in \mathbb{C}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Example. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$$

Example. We have shown $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z|<1$.

Making use of this we can compute the Taylor series for some other functions.

First, replace z by $-z$:

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}(-1)^n z^n \quad \text{for } |z|<1$$

Next, replace z by $1-z$. we get

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n \quad \text{for } |z-1|<1$$

which is the Taylor expansion of $\frac{1}{z}$ at $z_0=1$

Now we consider the Taylor expansion of $\frac{1}{1-z}$ at $z_0 = i$

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\cdot\frac{1}{1-\frac{z-i}{1-i}}$$

$$= \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{(1-i)^{n+1}}(z-i)^n \quad \text{for } |z-i|<\sqrt{2}$$

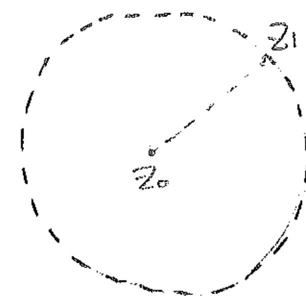
Example. $z^3 e^{2z} = z^3 \sum_{n=0}^{\infty} \frac{1}{n!}(2z)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{n+3}$

Remark. In the above examples, we indeed implicitly used the fact that the power series representation is unique. We'll show this fact later.

Now we are going to study some properties of power series.

Theorem. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z=z_1$ $(z_1 \neq z_0)$, then it is absolutely converged at any point on the open disk $|z-z_0|<|z_1-z_0|$.

Proof. If $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges,



$\lim_{n\to\infty} a_n(z_1-z_0)^n = 0$. So

there exists $M>0$ such that
$|a_n(z_1-z_0)^n| \leq M \quad \forall\, n \in \mathbb{N}$.

Then for any z such that $|z-z_0|<|z_1-z_0|$,

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n \leq M \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n$$

Since $\left|\frac{z-z_0}{z_1-z_0}\right|<1$, the series $\sum_{n=0}^{\infty} M\cdot\left|\frac{z-z_0}{z_1-z_0}\right|^n$ converges.

By Comparison Test, this indicates $\sum_{n=0}^{\infty} |a_n(z-z_0)|^n$ converges, i.e. $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent.

Definition. The greatest circle centred at z_0 such that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent at every point inside is called the circle of convergence of the series.

Corollary If a circle C is the circle of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then the series diverges at any point outside of C.

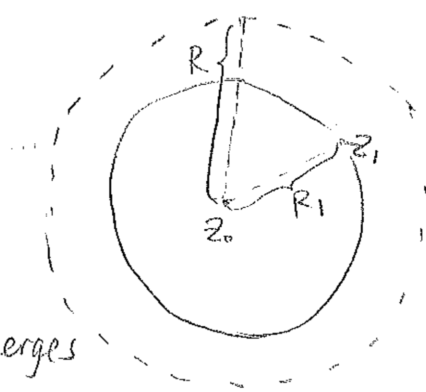
Proof. If the power series converges at some z outside of C, then by the theorem, the series is absolutely convergent inside a circle of radius $|z-z_0|$, larger than that of C, contradiction.

Definition. A series $\sum_{i=0}^{\infty} f_n(z)$ converges uniformly to a function $S(z)$ on a region R if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$N > N_0 \Rightarrow \quad \left|\sum_{i=0}^{N} f_n(z) - S(z)\right| < \varepsilon \quad \forall z \in R$$

Theorem If $\sum_{i=0}^{\infty} a_n(z-z_0)^n$ has circle of convergence $|z-z_0| = R$, $0 < R_1 < R$, then the series converges uniformly on the closed disk $|z-z_0| \le R_1$

Proof. Take a point z_1 such that $|z_1 - z_0| = R_1$. Since z_1 is inside the circle of convergence, the power series converges absolutely at z_1, i.e.



$$\sum_{n=0}^{\infty} |a_n(z_1-z_0)|^n = \sum_{n=0}^{\infty} |a_n R_1|^n \text{ converges}$$

For any $|z-z_0| \le R'$, $|a_n(z-z_0)|^n \le |a_n R_1|^n \quad \forall n \in \mathbb{N}$.

so we'll finish the proof by directly applying the following Lemma.

Lemma. $\sum_{n=0}^{\infty} C_n$ is a convergent real series such that $C_n \geq 0 \ \forall n \in \mathbb{N}$.
If $\sum_{n=0}^{\infty} f_n(z)$ is a series of complex functions such that $|f_n(z)| < C_n \ \forall n \in \mathbb{N}$ on a region R, then $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on R.

Proof. $\sum_{n=0}^{\infty} C_n$ converges, so for $\forall \varepsilon > 0$. $\exists N_\varepsilon \in \mathbb{N}$ such that

$$N > N_\varepsilon \Rightarrow \left|\sum_{n=N+1}^{\infty} C_n\right| = \sum_{n=N+1}^{\infty} C_n < \varepsilon \quad (\text{since } C_n \geq 0 \ \forall n \in \mathbb{N})$$

Since $\forall z \in R$, $|f_n(z)| \leq C_n$, and $\sum_{n=1}^{\infty} C_n$ Converges, so by Comparison Test $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely to some $S(z)$, i.e. there is a function $S(z)$ on R such that $S(z) = \sum_{n=0}^{\infty} f_n(z)$.

now for $N > N_\varepsilon \Rightarrow$,
$$|S(z) - \sum_{n=0}^{N} f_n(z)| = \left|\sum_{n=N+1}^{\infty} f_n(z)\right| \leq \sum_{n=N+1}^{\infty} |f_n(z)|$$
$$\leq \sum_{n=N+1}^{\infty} C_n < \varepsilon$$

so $\sum_{n=0}^{\infty} f_n(z)$ converges to $S(z)$ uniformly.

Theorem. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to a continuous function $S(z)$ inside its circle of convergence.

Proof. $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

If z_1 is inside the circle of convergence, take a circle C centred at z_0 such that z_1 is inside C and C is inside the circle of convergence. Then by the previous theorem, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges uniformly on the closed disk bounded by C.

For any $\varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ such that $N > N_\varepsilon \Rightarrow$

$$\left| S(z) - \sum_{n=0}^{N} a_n (z-z_0)^n \right| < \frac{\varepsilon}{3} \quad \forall\ z \text{ on or inside } C.$$

Note $\sum_{n=0}^{N} a_n (z-z_0)^n$ is a polynomial, so it's continuous at z_1, $\exists\ \delta > 0$ such that $|z-z_1| < \delta \Rightarrow$

$$\left| \sum_{n=0}^{N} a_n (z-z_0)^n - \sum_{n=0}^{N} a_n (z_1-z_0)^n \right| < \frac{\varepsilon}{3}.$$

Take $0 < \delta' < \delta$ such that $|z-z_1| < \delta' \Rightarrow z$ is inside C.

$$\begin{aligned} |S(z) - S(z_1)| &\le \left| S(z) - \sum_{n=0}^{N} a_n (z-z_0)^n \right| + \left| \sum_{n=0}^{N} a_n (z-z_0)^n - \sum_{n=0}^{N} a_n (z_1-z_0)^n \right| \\ &\quad + \left| \sum_{n=0}^{N} a_n (z_1-z_0)^n - S(z_1) \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

so $S(z)$ is continuous at z_1

Lemma. C is a contour interior to the circle of convergence of $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, and $g(z)$ is a function that is continuous on C. Then

$$\int_C g(z) S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z-z_0)^n\, dz$$

Proof. Write $S(z) = \sum_{n=0}^{N-1} a_n (z-z_0)^n + \rho_N(z)$

$$g(z) S(z) = \sum_{n=0}^{N-1} a_n g(z) (z-z_0)^n + g(z) \rho_N(z)$$

Note since $S(z)$ & $\sum_{n=1}^{N-1} a_n (z-z_0)^n$ are continuous on C. $\rho_N(z)$ is also continuous on C.

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz$$

In order to prove the Lemma, we only need to show $\lim_{N\to\infty} \int_C g(z)\rho_N(z)\,dz = 0$.

Since $g(z)$ is continuous on C, let $M = \max_{z\in C} |g(z)|$

By the uniformly convergence of $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$,

$\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon > 0$ such that

$$N > N_\varepsilon \Rightarrow \quad |\rho_N(z)| = \left|S(z) - \sum_{n=1}^{N-1} a_n (z-z_0)^n\right| < \frac{\varepsilon}{ML} \ \forall\ z \in C.$$

where L is the arclength of C.

So $N > N_\varepsilon \Rightarrow \left|\int_C g(z)\rho_N(z)\,dz\right| < M \cdot \frac{\varepsilon}{ML} \cdot L = \varepsilon$

we hence proved the Lemma.

Corollary. $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic inside the circle of convergence.

Proof. Pf. We take the constant function $g(z) \equiv 1$ in the previous Lemma, then for any closed contour C inside the circle of convergence,

$$\int_C S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,dz$$

Each term in the right side series is zero since $(z-z_0)^n$ is analytic $\forall\ n \in \mathbb{N}$.

We thus conclude $\int_C S(z)\,dz = 0$ for any closed contour C inside the circle of convergence, so $S(z)$ is analytic inside the circle of convergence.

Example. $f(z) = \begin{cases} \frac{\sin z}{z} & , z \neq 0 \\ 1 & , z = 0 \end{cases}$. We'll show $f(z)$ is entire:

so we need to show $f(z)$ is analytic at $z_0 = 0$

We know $\sin z = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

so for any $z \neq 0$,

$$f(z) = \frac{\sin z}{z} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{z^{2k}}{(2k+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

But $f(0) = 1$ satisfies the equation as well. so

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \quad \text{holds for all } z \in \mathbb{C}$$

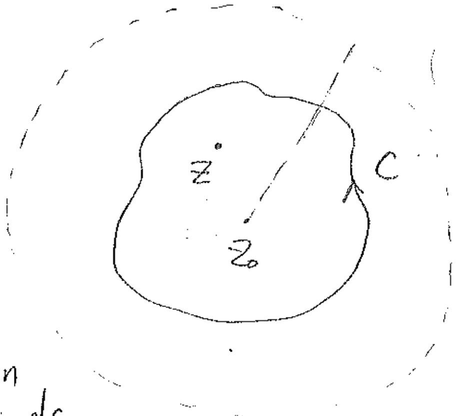
i.e. $f(z)$ equals to a series convergent everywhere, so $f(z)$ is entire (the circle of convergence has ∞-radius)

Theorem. $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. For any z inside the circle of convergence,

$$S'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1}$$

Proof. If z is inside the circle of convergence, take a positively oriented simple closed contour C such that C is inside the circle of convergence and z is inside C.

Let $g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s-z)^2}$ for $s \in C$



By the previous Lemma.

$$\int_C g(s)\, S(s)\, ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s-z_0)^n\, ds$$

$$\frac{1}{2\pi i}\int_C \frac{S(s)}{(s-z)^2}\, ds = \sum_{n=0}^{\infty} a_n \cdot \frac{1}{2\pi i}\int_C \frac{(s-z_0)^n}{(s-z)^2}\, ds$$

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{ds}(s-z_0)^n \Big|_{s=z}$$

$$S'(z) = \sum_{n=1}^{\infty} a_n \cdot n (z-z_0)^{n-1}$$

Example. We know $\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (|z-1|<1)$

so $-\frac{1}{z^2} = \left(\frac{1}{z}\right)' = \left(\sum_{n=0}^{\infty} (-1)^n (z-1)^n\right)'$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1}$$

so $\frac{1}{z^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n \quad (|z-1|<1)$

Theorem. If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for all points interior to some circle $|z-z_0| = R$, then $a_n = \frac{f^{(n)}(z_0)}{n!}$ i.e. the Taylor series expansion for an analytic function at z_0 is the unique series converging to f.

Proof. If $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ $(|z-z_0|<R)$, take C be a circle centred at z_0 with radius less than R.

Recall that we have proved $\int_C g(z) f(z)\,dz = \sum_{n=0} a_n \int_C g(z)(z-z_0)^n\,dz$

for any $g(z)$ continuous on C.

Now we choose $g_k(z) = \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^{k+1}}$, $k\in\mathbb{N}$ as $g(z)$.

So

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{n=0} a_n\cdot\frac{1}{2\pi i}\int_C (z-z_0)^{n-k-1}\,dz$$

$$\frac{f^{(k)}(z_0)}{k!} = a_k \quad \left(\text{since } \int_C (z-z_0)^{n-k-1}\,dz = \begin{cases} 0, & n\neq k \\ 2\pi i, & n=k \end{cases}\right)$$

Corollary If $\sum_{n=0}^{\infty} a_n(z-z_0)^n = 0$ for any point inside a circle $|z-z_0|=R$, then $a_n=0$ $\forall n\in\mathbb{N}$

LAURANT SERIES

Theorem. (Laurant's Theorem) f is analytic on an annular domain $R_1 < |z - z_0| < R_2$. C is any positively oriented simple closed contour around z_0 in that domain. Then for each z in $R_1 < |z - z_0| < R_2$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$ and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Another way of writing is

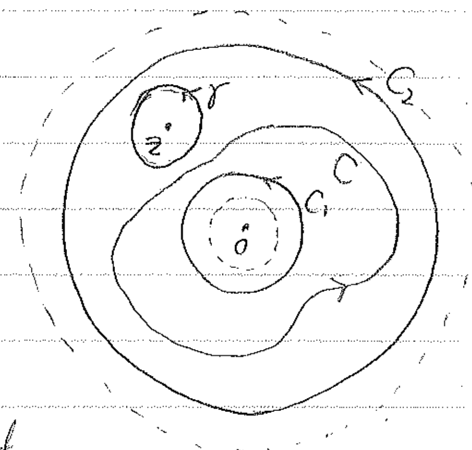
$$f(z) = \sum_{n=-\infty}^{+\infty} C_n (z - z_0)^n \quad \text{where } C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

And this series is called a Laurant Series.

Remark. By our previous discussion, we know that if f is analytic throughout $|z - z_0| < R_2$, then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$, so in this case the Laurant series for f on $R_1 < |z - z_0| < R_2$ has no negative term, and this agrees with the expression $C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$ that when $n < 0$, the integrand is analytic on and inside C in this case, so Cauchy-Goursat implies $C_n = 0$.

Proof. Similar to the proof for Taylor's Theorem, we'll first prove the case $z_0 = 0$.

Given a point z and a curve C on the domain $R_1 < |z - z_0| < R_2$, take $r_1, r_2 > 0$ such that $R_1 < r_1 < r_2 < R_2$ and z, C lie on ~~the~~ the smaller annulus $r_1 < |z| < r_2$. We denote C_1 and C_2 for the positively oriented circles $|z| = r_1$ and $|z| = r_2$



Take γ to be a small circle centred at z such that γ is in the smaller annulus $r_1 < |z| < r_2$ and γ is disjoint from C.

We apply the generalized Cauchy-Goursat Theorem to the multiply-connected domain inside C_2 and outside C_1 & γ, so we get

$$\int_{C_2} \frac{f(s)}{s-z}\,ds - \int_{C_1} \frac{f(s)}{s-z}\,ds - \int_{\gamma} \frac{f(s)}{s-z}\,ds = 0.$$

So $f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\,dz$

Recall that in the proof of the Taylor's Theorem, we get

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + z^N \cdot \frac{1}{(s-z)s^N} \quad \text{for } |z| < |s|.$$

So if we switch their roles, we get

$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \cdot s^N \frac{1}{(z-s)z^N} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \cdot \frac{1}{z^{n+1}} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}$$

$$= \sum_{n=1}^{N} \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}$$

We thus obtain

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\left(\sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + z^N\cdot\frac{1}{(s-z)s^N}\right)f(s)\,ds$$

$$+\frac{1}{2\pi i}\int_{C_1}\left(\sum_{n=1}^{N}\frac{1}{s^{-n+1}}\cdot\frac{1}{z^n} + \frac{1}{z^N}\cdot\frac{s^N}{z-s}\right)f(s)\,ds$$

$$=\left(\sum_{n=0}^{N-1}\frac{z^n}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right) + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}\,ds$$

$$+\left(\sum_{n=1}^{N}\frac{1}{z^n}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right) + \frac{1}{z^N}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{f(s)\cdot s^N}{z-s}\,ds$$

$$=\sum_{n=0}^{N-1}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right)z^n + \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right)\cdot\frac{1}{z^n}$$

$$+ R_N$$

where $R_N = \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}\,ds + \frac{1}{z^N}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{f(s)s^N}{z-s}\,ds$.

and it's not hard to show $\lim_{N\to\infty} R_N = 0$.

so $$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right)z^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right)\frac{1}{z^n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{n+1}}\,ds\right)z^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{-n+1}}\,ds\right)\cdot\frac{1}{z^n}$$

Next for the general case. let $g(z) = f(z+z_0)$

f analytic in $R_1<|z-z_0|<R_2 \Rightarrow g$ analytic in $R_1<|z|<R_2$.

so for the simple closed contour C in $R_1<|z-z_0|<R_2$, if it's given by $z(t)$, $(a\le t\le b)$. then Γ be the contour $z(t)-z_0$. then Γ is in $R_1<|z|<R_2$.

so by the special case at 0.

$$g(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \right) z^n + \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{g(s)}{s^{-n+1}} ds \right) \cdot \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s-z_0)} ds \right) (z-z_0)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int_{C} \frac{f(s)}{s^{-n+1}} ds \right) \frac{1}{(z-z_0)^n}$$

Example. $f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z} \cdot \frac{1}{1+z^2}$ is analytic on $0 < |z| < 1$.

(The singularities are 0, $\pm i$)

so on $0 < |z| < 1$,

$$f(z) = \frac{1}{z} \cdot \frac{1}{1+z^2} = \frac{1}{z} \left(\sum_{n=0}^{\infty} (-z^2)^n \right) = \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

Example. $f(z) = e^{\frac{1}{z}}$ is analytic on $0 < |z| < \infty$, so on this domain

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

Example. $f(z) = \frac{z+1}{z-1}$ is analytic on $0 < |z-1| < \infty$. so on this domain:

$$f(z) = \frac{z+1}{z-1} = \frac{1}{z-1} + 2$$

Also, $f(z)$ is analytic on $1 < |z| < \infty$. So

$$f(z) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right) \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{2}{z^n}$$

RESIDUES

Definition. If $f(z)$ is not analytic at z_0, but analytic at some point in every neighbourhood of it, we call z_0 a singular point of f

Definition. A singular point z_0 of f is isolated if there's a deleted ε-neighbourhood $0<|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic

Example. $z_0=0$ is a singular point for any branch of $f(z)=\log z$, but z_0 is not an isolated singular point since any neighbourhood $0<|z-z_0|<\varepsilon$ contains branch cut points

Example. $f(z)=\frac{1}{z}$ has an isolated singular point at $z_0=0$

Example. $f(z)=\dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$

The singular points are $z_0=0$ or $\frac{\pi}{z}=k\pi,\ k\in\mathbb{Z}$

i.e. $z_0=0$ or $z=\frac{1}{k},\ k\in\mathbb{Z}$

So we get $z_0=\frac{1}{k}, k\in\mathbb{Z}$ are isolated singular points.

$z_0=0$ is a singular point that is not isolated

Lemma. If there're only finitely many singular points for $f(z)$ on a domain U, then these singular points are all isolated.

Proof. Let $z_1,\dots,z_n$ be the set of singular points.

let $L_k=\min|z_k-z_i|$, then $f(z)$ is analytic on $0<|z-z_k|<L_k$, so z_k is isolated singular point.

Remark. ∞ is considered as an isolated singular point if $\exists\, R>0$ such that f is analytic on $|z|>R$.

Recall: If f is analytic on $\mathcal{U}: 0<|z-z_0|<R$ for some $R>0$ and $f(z)=\sum_{n=-\infty}^{+\infty} C_n (z-z_0)^n$ on $\mathcal{U}$, C is a positively oriented simple closed contour with z_0 in its interior, then

$$\int_C f(z)\,dz = 2\pi i\, C_{-1}$$

Definition. When z_0 is an isolated singularity of f, and the Laurant series expansion for f on $0<|z-z_0|<R$ for some $R>0$ is $f(z)=\sum_{n=-\infty}^{+\infty} C_n (z-z_0)^n$, then define the residue of f at z_0 to be $\operatorname*{Res}_{z=z_0}(f)=C_{-1}$

Example. Consider $\int_C \frac{e^z-1}{z^4}\,dz$ where C is the positively oriented circle $|z|=1$:

$$\frac{e^z-1}{z^4} = \frac{1}{z^4}(e^z-1) = \frac{1}{z^4}\left(\sum_{n=1}^{\infty}\frac{1}{n!}z^n\right) = \sum_{n=1}^{\infty}\frac{1}{n!}z^{n-4}$$

$$= \frac{1}{z^3} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z} + \cdots$$

so $\operatorname*{Res}_{z=0}(f) = \frac{1}{3!} = \frac{1}{6}$

We get $\int_C \frac{e^z-1}{z^4}\,dz = 2\pi i\cdot\frac{1}{6} = \frac{\pi i}{3}$

Example. $\int_C \frac{dz}{z(z-2)^5}$ where C is the positively oriented circle $|z-2|=1$.

Note this circle is inside $0<|z-2|<2$, on which $f(z)=\frac{1}{z(z-2)^5}$ is analytic. so the Laurant expansion on $0<|z-2|<2$ is:

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5}\cdot\frac{1}{z} = \frac{1}{(z-2)^5}\cdot\frac{1}{2+(z-2)}$$

$$= \frac{1}{(z-2)^5}\cdot\frac{1}{2}\cdot\frac{1}{1+\frac{z-2}{2}}$$

$$= \frac{1}{2(z-2)^5}\cdot\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\cdot(z-2)^{n-5}$$

we see $C_{-1} = \frac{(-1)^4}{2^{4+1}} = \frac{1}{32}$

so $\int_C f(z)\,dz = 2\pi i\cdot\frac{1}{32} = \frac{\pi i}{16}$

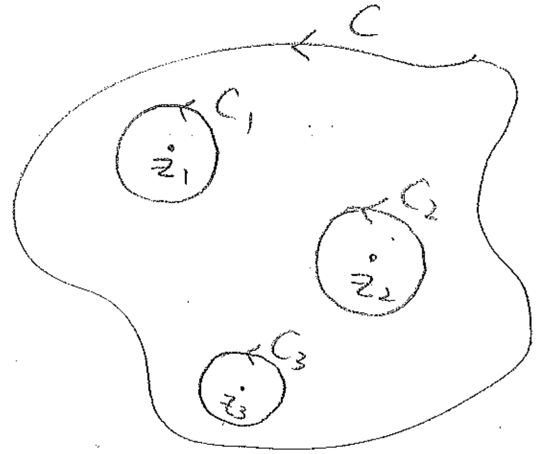
The above method of computing integrals by residue can be generalized by applying the multiply-connected version of Cauchy-Goursat Theorem, we get Cauchy's Residue Theorem:

Theorem. C is a simple closed contour positively oriented. If $f(z)$ is analytic on and inside C except for finitely many singular points $z_1,\dots,z_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}(f)$$

Proof. Take positively oriented circles C_k around z_k such that they're small enough to be inside C, and they bound n disjoint disks.

Then by the multiply connected version of Cauchy-Goursat Theorem,



$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

$$= \sum_{k=1}^{n} 2\pi i \cdot \operatorname*{Res}_{z=z_k}(f)$$

$$= 2\pi i \sum_{k=0}^{n} \operatorname*{Res}_{z=z_k}(f)$$

Example. Let's compute $\int_C \frac{4z-5}{z(z-1)}\,dz$, where C is given by the circle $|z|=2$, positively oriented.

Observe that inside C there're two singular points: 0 and 1.

$$\frac{4z-5}{z(z-1)} = \left(4-\frac{5}{z}\right)\cdot\frac{1}{z-1} = \left(4-\frac{5}{z}\right)(-1-z-z^2-\cdots) \qquad (0<|z|<1)$$

so $\operatorname*{Res}_{z=0}(f) = (-5)\times(-1) = 5$.

$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z-1}\cdot\frac{1}{z} = \left(4-\frac{1}{z-1}\right)\cdot\frac{1}{1-(1-z)} = \left(4-\frac{1}{z-1}\right)\cdot\left(1-(z-1)+\cdots\right)$$

$$(0<|z-1|<1)$$

so $\operatorname*{Res}_{z=1}(f) = -1$

We get $\int_C f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=0}(f) + \operatorname*{Res}_{z=1}(f)\right) = 2\pi i\,(5-1) = 8\pi i$

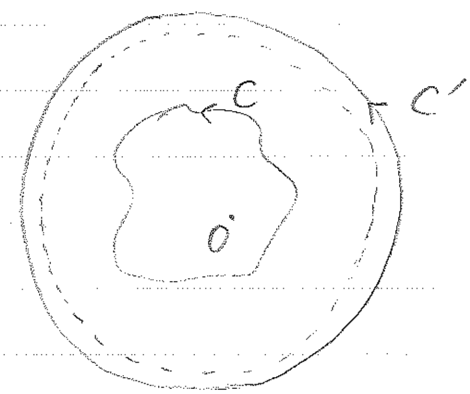
Theorem. If f is analytic everywhere in $\mathbb{C}$ except for a finite number of singular points interior to a positively oriented simple closed contour C, then:

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Proof. Take a positively oriented circle C': $|z|=R$ that encloses C in its interior.

then $f(z)$ is analytic on $|z|>R-\varepsilon$ which implies $f(\frac{1}{z})$ is analytic on $0<z<\frac{1}{R}$.

so 0 is an isolated singular point for $f(\frac{1}{z})$



Assume the Laurant series for $f(z)$ on $|z|>R$ is

$$f(z) = \sum_{-\infty}^{+\infty} C_n z^n, \quad \text{where } C_n = \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z^{n+1}}\,dz.$$

then for $0<z<\frac{1}{R}$,

$$f\left(\frac{1}{z}\right) = \sum_{-\infty}^{+\infty} C_n z^{-n}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{-\infty}^{+\infty} C_n z^{-n-2}$$

We see $\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f(\frac{1}{z})\right] = C_{-1}$

So $\int_C f(z)\,dz = \int_{C'} f(z)\,dz = 2\pi i\, C_{-1} = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f(\frac{1}{z})\right]$

Remark. We define $\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$

Example. $f(z) = \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$, C is the positively oriented circle $|z|=2$

If z_0 is a singular point of $f(z)$, then either $z_0=-1$ or $1+2z_0^4=0$

the latter implies $|z_0|^4 = |-\frac{1}{2}| = \frac{1}{2}$, so $|z_0| \le 1$.

We thus see all the singular points are inside C

so by the Theorem

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\cdot\frac{\left(\frac{1}{z}\right)^3\left(1-3\cdot\frac{1}{z}\right)}{\left(1+\frac{1}{z}\right)\left(1+2\cdot\left(\frac{1}{z}\right)^4\right)}$$

$$= \frac{1}{z^2}\cdot\frac{\frac{1}{z^3}\cdot\frac{z-3}{z}}{\frac{z+1}{z}\cdot\frac{2+z^4}{z^4}}$$

$$= \frac{1}{z}\cdot\frac{z-3}{(z+1)(2+z^4)}$$

Note $\dfrac{z-3}{(z+1)(2+z^4)}$ is analytic at $z_0=0$, so

$$\frac{1}{z}\cdot\frac{z-3}{(z+1)(2+z^4)} = \frac{1}{z}\cdot\left(\frac{0-3}{(0+1)(2+0^4)} + \text{positive power terms}\right)$$

So $\operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2}$

we conclude $\int_C f(z)\,dz = 2\pi i\cdot\left(-\frac{3}{2}\right) = -3\pi i$

CLASSIFICATION OF ISOLATED SINGULARITIES

If z_0 is an isolated singular point for a function f, we know that in a deleted neighbourhood $0<|z-z_0|<\varepsilon$, we can express $f(z)$ in terms of Laurant Series:

$$f(z)=\sum_{n=-\infty}^{+\infty} C_n (z-z_0)^n$$

Definition Under the above assumption, we say z_0 is a:

(i) Removable Singular Point if $C_n=0 \ \forall\ n<0$

(ii) Pole if $C_n \neq 0$ for finitely many but at least one $n<0$

(iii) Essential Singular Point if $C_n \neq 0$ for infinitely many $n<0$

Now we will discuss each of the three cases.

Proposition If z_0 is a removable singularity, then the function

$$g(z)=\begin{cases} C_0, & z=z_0 \\ f(z), & z\neq z_0 \end{cases}$$

is analytic at z_0.

Proof. Since $g(z)=f(z)=\sum_{n=-\infty}^{+\infty} C_n(z-z_0)^n=\sum_{n=0}^{\infty} C_n(z-z_0)^n$ for $0<|z-z_0|<\varepsilon$ for z_0 a removable singularity, and by our construction, $g(z_0)=C_0$ also agrees with the power series, we get

$$g(z)=\sum_{n=0}^{\infty} C_n(z-z_0)^n \text{ for all } |z-z_0|<\varepsilon.$$

So $g(z)$ is analytic on $|z-z_0|<\varepsilon$, hence analytic at z_0

Remark. In other words, z_0 is a removable singularity means if we adjust the value $f(z_0)$, we can make z_0 a regular point (i.e not a singularity)

Example $f(z) = \begin{cases} 1, & z=0 \\ z, & z \neq 0 \end{cases}$ has a removable singularity at $z_0 = 0$.

If we adjust $f(0)$ to be 0, then $z_0 = 0$

Exercise $f(z) = \frac{\sin z}{z}$ has a removable singular point $z_0 = 0$

Definition. If z_0 is a pole for f and $C_{-m} \neq 0$, $C_{-n} = 0 \ \forall \ n > m$, we say z_0 is a pole of order m

By definition, if z_0 is a pole of order m, then the Laurant expansion near z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} b_n \frac{1}{(z-z_0)^n} \quad (0 < |z-z_0| < \varepsilon)$$

Definition. A pole of order 1 is called a simple pole.

Example. $f(z) = \frac{1}{z}$ has a simple pole $z_0 = 0$

$f(z) = \frac{1}{z^2} + \frac{1}{z} + z^3$ has a pole of order 2 at $z_0 = 0$

Example. $f(z) = e^{\frac{1}{z}}$ has an essential singular point at $z_0 = 0$ since the Laurant expansion around 0 is

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n} \quad (0 < |z| < \infty)$$

Theorem. z_0 is an isolated singular point of f. The following are equivalent:

(i) z_0 is a pole of order m of f

(ii) If f is analytic on $0<|z-z_0|<R$, then there exists a function ϕ analytic on $|z-z_0|<R$ such that $\phi(z_0)\neq 0$ and $f(z)=\dfrac{\phi(z)}{(z-z_0)^m}$

Proof. (i) $\Rightarrow$ (ii): If z_0 is a pole of order m, and f is analytic on $0<|z-z_0|<R$, the Laurent series on $0<|z-z_0|<R$ is

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\cdots+\frac{b_m}{(z-z_0)^m}\quad(0<|z-z_0|<R)$$

with $b_m\neq 0$.

Define $\phi(z)=\begin{cases} b_m & \text{if } z=z_0 \\ (z-z_0)^m f(z) & \text{if } 0<|z-z_0|<R. \end{cases}$

We see $\phi(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^{n+m}+b_1(z-z_0)^{m-1}+\cdots+b_{m-1}(z-z_0)+b_m$ for $0<|z-z_0|<R$, and this equality also holds at z_0.

So $\phi(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^{n+m}+b_1(z-z_0)^{m-1}+\cdots+b_{m-1}(z-z_0)+b_m$ is a power series convergent throughout $|z|<R$, we get $\phi(z)$ is analytic on $|z|<R$ with $\phi(z_0)=b_m\neq 0$, and $f(z)=\dfrac{\phi(z)}{(z-z_0)^m}$ on $0<|z-z_0|<R$

(ii) $\Rightarrow$ (i): If $f(z)=\dfrac{\phi(z)}{(z-z_0)^m}$ on $0<|z-z_0|<R$.

· since $\phi(z)$ is analytic on $|z-z_0|<R$,

$$\phi(z)=\sum_{n=0}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n=\phi(z_0)+\sum_{n=1}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \text{ on } |z-z_0|<R$$

So $f(z)=\sum_{n=0}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$

$$=\frac{\phi(z_0)}{(z-z_0)^m}+\sum_{n=1}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \quad (0<|z-z_0|<R)$$

Recall that the assumption is $\phi(z_0)\neq 0$, we see f has a pole of order m at z_0.

Corollary. If $f(z)=\frac{\phi(z)}{(z-z_0)^m}$ on $0<|z-z_0|<R$ for some ϕ analytic on $|z-z_0|<R$ with $\phi(z_0)\neq 0$, then

$$\operatorname*{Res}_{z=z_0}(f)=\frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Proof. By the proof of the Theorem, we see

$$f(z)=\frac{\phi(z_0)}{(z-z_0)^m}+\sum_{n=1}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \text{ on } 0<|z-z_0|<R$$

So $\operatorname*{Res}_{z=z_0}(f)=\frac{\phi^{(m-1)}(z_0)}{(m-1)!}$

Example. When the pole at z_0 is a simple pole, $m=1$ so the above formula reduces to

$$\operatorname*{Res}_{z=z_0}(f)=\phi(z_0)$$

Example. $f(z) = \dfrac{z+4}{z^2+1}$ has an isolated singularity at $z_0 = i$

$f(z) = \dfrac{\phi(z)}{z-i}$, where $\phi(z) = \dfrac{z+4}{z+i}$ is analytic

at $z_0 = i$, and $\phi(i) \neq 0$, so

$$\operatorname*{Res}_{z=i}(f) = \phi(i) = \frac{4+i}{2i}$$

Example. $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ has a pole at $z_0 = i$

$f(z) = \dfrac{\phi(z)}{(z-i)^3}$, $\phi(z) = z^3+2z$ is analytic at $z_0 = i$,

and $\phi(i) \neq 0$. So the pole $z_0 = i$ is of order 3

$$\operatorname*{Res}_{z=i}(f) = \frac{\phi^{(3-1)}(i)}{(3-1)!} = \frac{\phi''(i)}{2!} = 3i$$

Example. $f(z) = \dfrac{(\log z)^3}{z^2+1}$, $\log z$ is the branch $0 < \theta < 2\pi$.

$f(z) = \dfrac{\phi(z)}{z-i}$, where $\phi(z) = \dfrac{(\log z)^3}{z+i}$.

$\phi(z)$ is analytic at $z_0 = i$ and $\phi(i) = \dfrac{(\frac{\pi}{2}i)^3}{2i} = -\dfrac{\pi^3}{16} \neq 0$.

So $f(z)$ has a simple pole at $z_0 = i$, and

$$\operatorname*{Res}_{z=i}(f) = \phi(i) = -\frac{\pi^3}{16}$$

ZEROS OF ANALYTIC FUNCTIONS

Definition. If f is analytic at z_0, and there exists $m \in \mathbb{N}$ such that $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$, $f^{(m)}(z_0) \neq 0$, then we say z_0 is a zero of order m of f.

Example: $f(z) = (z-1)^2$ has a zero of order 2 at $z_0 = 1$.

Theorem. f is analytic at z_0, then the following are equivalent:

(i) f has zero of order m at z_0

(ii) $\exists\, g(z)$, analytic and nozero at z_0 such that

$$f(z) = (z-z_0)^m g(z).$$

Proof. (i) $\Rightarrow$ (ii). If z_0 is a zero of order m, the Taylor series expansion of f at z_0 is:

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n,$$

where $|z-z_0| < \varepsilon$ for some $\varepsilon > 0$ such that f is analytic on $|z-z_0| < \varepsilon$

Let $g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n$, since it converges on $|z-z_0| < \varepsilon$, so it's analytic on $|z-z_0| < \varepsilon$, and

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

(ii) $\Rightarrow$ (i). If $f(z) = (z-z_0)^m g(z)$ for some $g(z)$ analytic at z_0 and $g(z_0) \neq 0$, then:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{on} \quad |z-z_0|<\varepsilon, \quad g(z_0)\neq 0.$$

so $f(z) = (z-z_0)^m g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n+m}$.

which is the Taylor expansion for f at z_0, we see

$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$; $f^{(m)}(z_0) \neq 0$.

so z_0 is a zero of order m.

Example. $f(z) = z^3 - 1$, $f(1) = 0$, $f'(1) = 3 \neq 0$, so $z_0 = 1$ is a zero of order 1.

$f(z) = (z-1)g(z)$ where $g(z) = z^2 + z + 1$

Proposition. If f is analytic at z_0, and f is not constant on any neighbourhood of z_0, then there is a neighbourhood $0<|z-z_0|<\varepsilon$ such that $f(z) \neq f(z_0)$ on this neighbourhood

Proof. Let $F(z) = f(z) - f(z_0)$. then $F(z_0) = 0$ and F is analytic at z_0, F not constant on any neighbourhood of z_0.,
so we see not all $F^{(n)}(z_0) \neq 0$, otherwise the Taylor expansion of F at z_0 indicates $F(z) \equiv 0$ on a neighbourhood of z_0

Let the order of z_0 be m, then $\exists\, g$ such that $g(z)$ is analytic at z_0, $g(z_0) \neq 0$, and

$$F(z) = (z-z_0)^m g(z)$$

Note that $g(z_0) \neq 0$. so by continuity, $g(z) \neq 0$ on $|z-z_0|<\varepsilon$ for some $\varepsilon > 0$., so $F(z) \neq 0$ on $0<|z-z_0|<\varepsilon$, i.e $f(z) \neq f(z_0)$

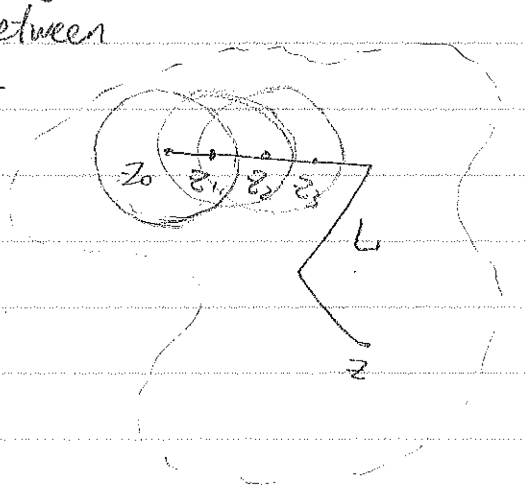
Corollary. If f is analytic throughout a neighbourhood N_0 of z_0, and there is a sequence $\{z_n\}$ such that $\lim_{n\to\infty} z_n = z_0$, $\forall n \in \mathbb{N}$, $z_n \neq z$ and $f(z_n)=0$, then $f(z) \equiv 0$ on N_0.

Proof. First, we see there exits neighbourhood N of z_0 such that $f(z) \equiv f(z_0) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = 0$, otherwise z_0 will be an isolated zero, contradict to $\lim_{n\to\infty} z_n = z_0$ & $f(z_n)=0$

So the Taylor series expansion of f at z_0 is $f(z)\equiv 0$. And $f(z)$ is also analytic on $N_0 \ni z_0$, the same taylor series expansion $f(z)\equiv 0$ holds on N_0 as well.

Theorem (Coincidence Principle) A function f is analytic on a domain D, and $\{z_n\}$ is a sequence in D with $\lim_{n\to\infty} z_n = z_0 \in D$ and $z_n \neq z_0$ $\forall n \in \mathbb{N}$. If $f(z_n)=0$ $\forall n\in\mathbb{N}$, then $f(z)\equiv 0$ on D

Proof For any $z\in D$, connect z_0 & z by a polygonal path L. Let d be the shortest distance between ∂D and L. Along L we pick points $z_0 = S_0$, $S_1, S_2, \dots, S_n$ such that $|S_k - S_{k-1}| < d$, and consider the balls $B_k = B(S_k, d)$ so each $S_{k+1} \in B_k$



On B_0, by the previous corollary, we see $f\equiv 0$ on B_0, so it's zero on $L \cap B_0$, which tells us z_1 is a limit of a sequence of zeros, so by the previous corollary again, we get $f \equiv 0$ on B_1. Continue this argument along L, we can finish the proof

Corollary. f and g are functions analytic on a domain D. $\{z_n\}$ is a sequence in D such that $\lim_{n\to\infty} z_n = z_0 \in D$, $z_n \neq z_0 \ \forall n \in \mathbb{N}$. If $f(z_n) = 0 \ \forall n \in \mathbb{N}$, then $f(z) \equiv 0$ on D

Proof. Take $f(z) - g(z)$ as the function in the Theorem

Next we are discussing the relations between zeros & poles

Theorem. If $p(z)$ and $q(z)$ are analytic at z_0, and $p(z_0) \neq 0$. $q(z)$ has a zero of order m at z_0, then:

$\frac{p(z)}{q(z)}$ has a pole of order m at z_0

Proof $q(z)$ has a zero of order m at z_0 implies

$$q(z) = (z - z_0)^m g(z)$$

for some $g(z)$ analytic at z_0 and $g(z_0) \neq 0$

Then $\frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)^m g(z)} = \frac{\frac{p(z)}{g(z)}}{(z-z_0)^m}$, Note that

$\frac{p(z)}{g(z)}$ is analytic at z_0 and $\frac{p(z_0)}{g(z_0)} \neq 0$. so

$\frac{p(z)}{q(z)}$ has a pole of order m at z_0.

Example. $\frac{1}{1-\cos z}$ has a pole of order 2 at $z_0 = 0$ since

we can let $p(z) \equiv 1$, $q(z) = 1 - \cos z$ in the above theorem:

$p(z) \equiv 1$ is analytic & nonzero at $z_0 = 0$.

$q(z) = 1 - \cos z$ has a zero of order 2 at $z_0 = 0$

Theorem. $p(z)$ and $q(z)$ are analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$, then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof. $q(z_0) = 0$, $q'(z_0) \neq 0 \Rightarrow q$ has a zero of order one at z_0. by the previous theorem, $\frac{p(z)}{q(z)}$ has a simple pole at z_0

Let $q(z) = (z-z_0) g(z)$, $g(z)$ is analytic at z_0, $g(z_0) \neq 0$.

Then $\frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)g(z)} = \frac{\frac{p(z)}{g(z)}}{z-z_0}$

$$\operatorname*{Res}_{z=z_0} \left(\frac{p(z)}{q(z)}\right) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)} \quad \left(\text{since } q'(z) = g(z) + (z-z_0)g'(z)\right)$$

Example. $f(z) = \frac{z}{z^4+4}$. $z_0 = \sqrt{2} e^{\frac{\pi}{4} i}$ is a pole of f.

let $p(z) = z$, $q(z) = z^4 + 4$, we see $p(z_0) \neq 0$, $q(z_0) = 0$ but $q'(z_0) = 4 z_0^3 \neq 0$, so by the theorem we get

$$\operatorname*{Res}_{z \to z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4 z_0^3} = \frac{1}{4 z_0^2} = \frac{1}{8} \cdot e^{-\frac{\pi}{2} i} = -\frac{i}{8}$$

BEHAVIOUR NEAR SINGULARITIES

Theorem. If z_0 is a removable singular point of z_0, then f is bounded and analytic on some deleted neighbourhood $0 < |z - z_0| < \varepsilon$.

Proof. There exists $F(z)$ such that $F(z) = f(z)$ for $z \neq z_0$ and $F(z)$ is analytic at z_0.

Since z_0 is an isolated singular point, we can find some $\varepsilon > 0$ such that $F(z) = F(z_0)$ is analytic on $0 < |z - z_0| < \varepsilon$ so $F(z)$ is analytic on $0 < |z - z_0| < \varepsilon$, hence continuous, $F(z)$ is therefore bounded on $|z - z_0| \leq \frac{\varepsilon}{2}$, a closed disk so also bounded on the subset $0 < |z - z_0| < \frac{\varepsilon}{2}$, which implies $f(z)$ is also bounded on $0 < |z - z_0| < \frac{\varepsilon}{2}$.

Theorem. (Riemann's Theorem)
Suppose f is bounded and analytic on $0 < |z - z_0| < \varepsilon$. If f is not analytic at z_0, then z_0 is a removable singular point.

Proof.
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{b=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (0 < |z - z_0| < \varepsilon)$$

Let $0 < r < \varepsilon$, and C be the positively oriented contour $|z - z_0| = r$.

Assume f is bounded by M on $0 < |z - z_0| < \varepsilon$, We know $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

$|b_n| \leq \frac{1}{2\pi} \cdot M \cdot \frac{1}{r^{-n+1}} \cdot 2\pi r = M \cdot r^n$

Since the above holds for any $0<r<\varepsilon$, letting $r \to 0$, we get $|b_n| = 0 \Rightarrow b_n = 0$

Theorem. (Casorati-Weierstrass Theorem)
z_0 is an essential singular point of f. For any $w_0 \in \mathbb{C}$, any $\varepsilon > 0$ and any $\delta > 0$, there exists $0<|z-z_0|<\delta$ such that $|f(z) - w_0| < \varepsilon$.

Proof Suppose we can find $w_0 \in \mathbb{C}$, $\varepsilon > 0$ and $\delta > 0$ such that $\forall\ 0<|z-z_0|<\delta$, $|f(z)-w_0| \geq \varepsilon$.

By taking even small δ if necessary, we can assume f is analytic on $0<|z-z_0|<\delta$.

Define $g(z) = \frac{1}{f(z)-w_0}$ $\quad (0<|z-z_0|<\delta)$

$|g(z)| = \frac{1}{|f(z)-w_0|} \leq \frac{1}{\varepsilon}$ on $0<|z-z_0|<\delta$, so

$g(z)$ is bounded and analytic on $0<|z-z_0|<\delta$, applying Riemann's Theorem, z_0 is a removable singular point of g. So we can extend $g(z)$ to $G(z)$, which is analytic on $|z-z_0|<\delta$.

If $G(z_0) \neq 0$, define $F(z) = \frac{1}{G(z)} + w_0$ on $|z - z_0| < \delta$.

When $0 < |z - z_0| < \delta$, observe that

$$F(z) = \frac{1}{g(z)} + w_0 \Rightarrow g(z) = \frac{1}{F(z) - w_0},$$

So $F(z) = f(z)$ on $0 < |z - z_0| < \delta$, and $F(z)$ is analytic at z_0, we see z_0 is a removable singular point of $f(z)$, contradiction.

If $G(z_0) = 0$, since $G(z)$ is not constant, z_0 is a zero of some finite order m of G.

$$\therefore f(z) = \frac{1}{g(z)} + w_0 = \frac{1 + w_0 G(z)}{G(z)}$$

$1 + w_0 G(z_0) = 1 \neq 0$, and $G(z)$ has zero of order m at z_0, so $f(z)$ has a pole of order m at z_0, contradiction.

Theorem. If z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$

Proof. Assume z_0 is a pole of order m, then

$f(z) = \frac{\phi(z)}{(z - z_0)^m}$ for some $\phi(z)$ analytic at z_0 and $\phi(z_0) \neq 0$

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{0}{\phi(z_0)} = 0$$

since $\phi(z)$ is continuous at z_0.

We get $\lim_{z \to z_0} f(z) = \infty$

IMPROPER INTEGRALS

Definition. $f : \mathbb{R} \to \mathbb{R}$ is a real function. The improper integral $\int_{-\infty}^{+\infty} f(x)\,dx$ is defined to be the number

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R_1 \to -\infty} \int_{R_1}^{0} f(x)\,dx + \lim_{R_2 \to +\infty} \int_{0}^{R_2} f(x)\,dx$$

if both limits converge.

Definition. $f : \mathbb{R} \to \mathbb{R}$ is a real function. The Cauchy Principal Value of the improper integral is defined to be

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R \to +\infty} \int_{-R}^{R} f(x)\,dx$$

if the limit converges.

Remark. Note that the existence of $\int_{-\infty}^{+\infty} f(x)\,dx$ implies the convergence of $\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$, but the reverse is not always true.

For example $\int_{-\infty}^{+\infty} x\,dx$ diverges, but $\text{P.V.} \int_{-\infty}^{+\infty} x\,dx = 0$

Lemma. If $f : \mathbb{R} \to \mathbb{R}$ is an even function, and $\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$ converges, then $\int_{-\infty}^{+\infty} f(x)\,dx$ converges and

$$\int_{-\infty}^{+\infty} f(x)\,dx = \text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$$

Moreover,

$$\int_{-\infty}^{0} f(x)\,dx = \int_{0}^{+\infty} f(x)\,dx = \frac{1}{2}\, \text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$$

A useful application of residues is to compute improper integrals.

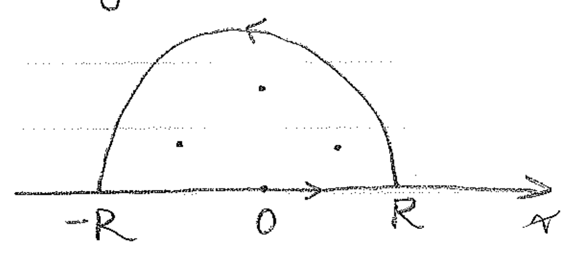
Example. $\displaystyle\int_0^{\infty} \frac{1}{x^6+1}\, dx$

This is an even function, so we can compute
$\displaystyle P.V. \int_{-\infty}^{+\infty} \frac{1}{x^6+1}\, dx$ first.

Consider the complex function $f(z) = \dfrac{1}{z^6+1}$

The singular points of $f(z)$ are the zeros of z^6+1,
which are $e^{\frac{\pi}{6}i}$, $e^{\frac{\pi}{2}i}$, $e^{\frac{5\pi}{6}i}$, $e^{\frac{7}{6}\pi i}$, $e^{\frac{3}{2}\pi i}$, $e^{\frac{11}{6}\pi i}$

Construct a semicircle with centre at 0, radius $R>1$,
C_R given by $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Let L_R be the line segment $[-R, R]$, pointing to right.



Then:

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = \Big[\operatorname*{Res}_{z=e^{\frac{\pi}{6}i}}(f) + \operatorname*{Res}_{z=e^{\frac{\pi}{2}i}}(f) + \operatorname*{Res}_{z=e^{\frac{5}{6}\pi i}}(f)\Big]\cdot 2\pi i$$

Note $e^{\frac{\pi}{6}i}$, $e^{\frac{\pi}{2}i}$, $e^{\frac{5}{6}\pi i}$ are zeros of order 1 for z^6+1,

So $\displaystyle\operatorname*{Res}_{z=e^{\frac{\pi}{6}i}}(f) = \frac{1}{6(e^{\frac{\pi}{6}i})^5} = -\frac{1}{6}e^{\frac{\pi}{6}i}$

$\displaystyle\operatorname*{Res}_{z=e^{\frac{\pi}{2}i}}(f) = \frac{1}{6(e^{\frac{\pi}{2}i})^5} = -\frac{1}{6}e^{\frac{\pi}{2}i}$

$\displaystyle\operatorname*{Res}_{z=e^{\frac{5}{6}\pi i}}(f) = \frac{1}{6(e^{\frac{5}{6}\pi i})^5} = -\frac{1}{6}e^{\frac{5}{6}\pi i}$

We get

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = -\frac{1}{6}\left(e^{\frac{\pi}{6}i} + e^{\frac{\pi}{2}i} + e^{\frac{5}{6}\pi i}\right)\cdot 2\pi i$$

$$= -\frac{1}{6}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i + i + \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right)\cdot 2\pi i$$

$$= -\frac{1}{6}\times 2i \times 2\pi i$$

$$= \frac{2}{3}\pi$$

Note $\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{R^6-1}\cdot \pi R \longrightarrow 0$ as $R \to +\infty$.

We get $\lim_{R\to+\infty}\int_{C_R} f(z)\,dz = 0$

So $P.V.\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to+\infty}\int_{L_R} f(z)\,dz = \frac{2}{3}\pi$

$$\int_0^{+\infty} f(x)\,dx = \frac{1}{2}\times\frac{2}{3}\pi = \frac{\pi}{3}$$

Example $\int_0^{+\infty}\frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{5\pi}{16e^4}$:

Let $f(z) = \frac{e^{i2z}}{(z^2+4)^2}$

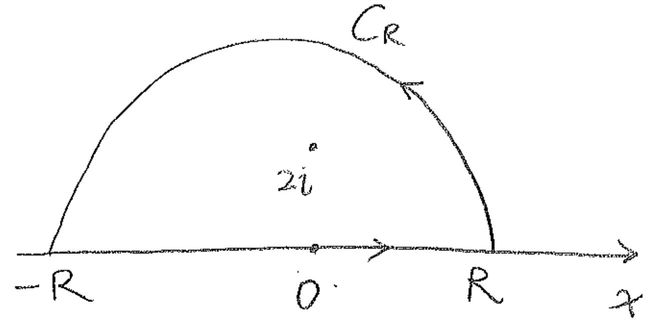
The singular points of $f(z)$ are zeros of $(z^2+4)^2$: $2i$ & $-2i$

Let $R>2$, C_R the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$.
L_R the line segment $[-R, R]$, pointing to the right

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=2i}(f)$$

$$f(z) = \frac{\dfrac{e^{i\cdot 2z}}{(z+2i)^2}}{(z-2i)^2}.$$

Observe that $\phi(z) = \dfrac{e^{i\cdot 2z}}{(z+2i)^2}$ is analytic at $z=2i$ and $\phi(2i) \neq 0$.



$$\phi'(z) = \frac{2ie^{i\cdot 2z}(z+2i)^2 - 2e^{i\cdot 2z}(z+2i)}{(z+2i)^4}$$

$$\phi'(2i) = \frac{2i\cdot e^{-4}\cdot(4i)^2 - 2e^{-4}(4i)}{(4i)^4}$$

$$= \frac{-2\times 4^2 i - 8i}{4^4 e^4}$$

$$= \frac{-5i}{32e^4}$$

So $\displaystyle \operatorname*{Res}_{z=2i}(f) = \phi'(2i) = \frac{-5i}{32e^4}$

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i\cdot\frac{-5i}{32e^4} = \frac{5\pi}{16e^4}$$

If z is on C_R, we see

$$|z^2+4| \geq |z^2| - 4 = |z|^2 - 4 = R^2 - 4.$$

$|e^{i2z}| = |e^{i\cdot 2(x+iy)}| = e^{-2y} \leq 1$ since $y \geq 0$ on C_R.

So $\displaystyle \left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{(R^2-4)^2}\cdot\pi R \longrightarrow 0$ as $R \longrightarrow +\infty$

we get $\displaystyle \lim_{R\to+\infty}\int_{L_R} f(z)\,dz = \frac{5\pi}{16e^4}$, i.e.

$$\text{P.V.}\int_{-\infty}^{+\infty}\frac{\cos 2z}{(z^2+4)^2}\,dx + i\cdot\text{P.V}\int_{-\infty}^{+\infty}\frac{\sin 2z}{(z^2+4)^2}\,dx = \frac{5\pi}{16e^4}.$$

Lemma. (Jordan's Lemma)

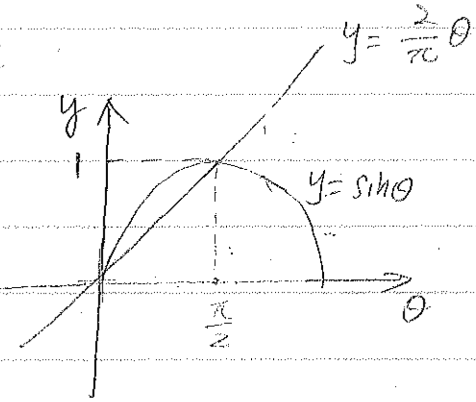
$f(z)$ is analytic at all points in the upper half plane $y \geq 0$ exterior to a circle $|z| = R_0$, and C_R is the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, $(R > R_0)$. If $|f(z)| < M_R$ $\forall z \in C_R$, and $\lim_{R \to +\infty} M_R = 0$, Then:

$$\lim_{R \to +\infty} \int_{C_R} f(z) e^{iaz} dz = 0 \qquad \forall\, a > 0.$$

Proof. Since $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$

$R > 0$, then $R\sin\theta \geq R\,\frac{2\theta}{\pi}$

$$e^{-R\sin\theta} \leq e^{-\frac{2R}{\pi}\theta} \quad \text{for } 0 \leq \theta \leq \tfrac{\pi}{2}.$$



This implies

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta} d\theta = -\frac{\pi}{2R} e^{-\frac{2R}{\pi}\theta}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2R}(e^{-R} - 1)$$

$$= \frac{\pi}{2R}(1 - e^{-R})$$

$$< \frac{\pi}{2R}$$

We get the Jordan's Inequality:

$$\int_0^{\pi} e^{-R\sin\theta} d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} < \frac{\pi}{R} \quad \text{for } R > 0$$

$$\int_{C_R} f(z) e^{iaz} dz = \int_0^{\pi} f(Re^{i\theta}) e^{iaRe^{i\theta}} \cdot Rie^{i\theta} d\theta$$

Note that $|f(Re^{i\theta})| \le M_R$,

$$|e^{iaRe^{i\theta}}| = |e^{iaR(\cos\theta + i\sin\theta)}| = e^{-aR\sin\theta}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le M_R \cdot R \int_0^{\pi} e^{-aR\sin\theta}\,d\theta < M_R \cdot R \cdot \frac{\pi}{aR} = \frac{\pi}{a} M_R$$

Since $\lim_{R\to+\infty} M_R = 0$, we conclude $\lim_{R\to+\infty} \int_{C_R} f(z)\,dz = 0$

This Lemma has applications in some improper integrals involving sine and cosine functions.

Example $\displaystyle\int_0^{\infty} \frac{x \sin 2x}{x^2+3}\,dx$

Let $f(z) = \dfrac{z}{z^2+3} \cdot e^{i2z} = \dfrac{z}{(z+\sqrt{3}i)(z-\sqrt{3}i)} \cdot e^{i \cdot 2z}$

The singular points of $f(z)$ are $z = \pm\sqrt{3}i$.

$$\operatorname*{Res}_{z=\sqrt{3}i}(f) = \frac{\sqrt{3}i}{(\sqrt{3}i+\sqrt{3}i)} \cdot e^{i2\sqrt{3}i} = \frac{1}{2e^{2\sqrt{3}}}$$

Let $R > \sqrt{3}$, C_R be $z(\theta) = \sqrt{3}e^{i\theta}$, $0 \le \theta \le \pi$.

$L_R = [-R, R]$, pointing to the right.

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=\sqrt{3}i}(f) = \frac{\pi i}{e^{2\sqrt{3}}}$$

For each $R > \sqrt{3}$, $\left|\dfrac{z}{z^2+3}\right| \le \dfrac{R}{R^2-3}$ on C_R,

and $\lim_{R\to\infty} \dfrac{R}{R^2-3} = 0$.

By Jordan's Lemma.

$$\lim_{R\to\infty}\int_{C_R}\frac{z}{z^2+3}\cdot e^{i\cdot 2z}\,dz=0$$

so $$\lim_{R\to\infty}\int_{L_R}f(z)\,dz=\frac{\pi i}{e^{2\sqrt{3}}}$$

$$P.V.\int_{-\infty}^{+\infty}\frac{x\cos 2x}{x^2+3}dx+i\,P.V.\int_{-\infty}^{+\infty}\frac{x\sin 2x}{x^2+3}dx=\frac{\pi}{e^{2\sqrt{3}}}i$$

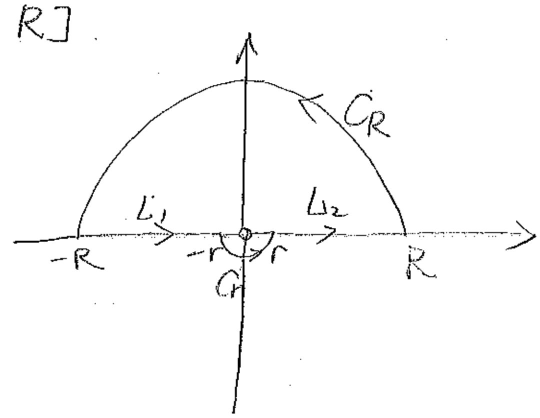
we see $$P.V.\int_{-\infty}^{+\infty}\frac{x\sin 2x}{x^2+3}dx=\frac{\pi}{e^{2\sqrt{3}}}$$

so $$\int_0^{+\infty}\frac{x\sin 2x}{x^2+3}dx=\frac{1}{2}\cdot\frac{\pi}{e^{2\sqrt{3}}}=\frac{\pi}{2e^{2\sqrt{3}}}$$

Example. Evaluate the Dirichlet's Integral. $\int_0^{\infty}\frac{\sin x}{x}dx$

Let $f(z)=\frac{e^{iz}}{z}$, note $f(z)$ has a simple pole at $z=0$.

Let $0<r<R$, C_R be the semicircle $z=Re^{i\theta}$, $0\le\theta\le\pi$.
C_r be the semicircle $z=re^{i\theta}$, $\pi\le\theta\le 2\pi$
L_1 be $[-R,-r]$, L_2 be $[r, R]$.



$$(*)\ \int_{C_r}f(z)\,dz+\int_{L_2}f(z)\,dz+\int_{C_R}f(z)\,dz+\int_{L_1}f(z)\,dz$$

$$=2\pi i\cdot\operatorname*{Res}_{z=0}f(z)$$

$$=2\pi i$$

By Jordan's Lemma, if $R \to \infty$

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0. \quad \text{since on } C_R,\ \left|\frac{1}{z}\right| = \frac{1}{R} \to 0$$

$$\int_{L_1} f(z)\, dz = \int_{-R}^{-r} \frac{e^{ix}}{x} dx = \int_{-R}^{-r} \frac{\cos x}{x} dx + i\int_{-R}^{-r} \frac{\sin x}{x} dx$$

$$\int_{L_2} f(z)\, dz = \int_{r}^{R} \frac{e^{ix}}{x} dx = \int_{r}^{R} \frac{\cos x}{x} dx + i\int_{r}^{R} \frac{\sin x}{x} dx$$

So $\int_{L_1} f(z)\, dz + \int_{L_2} f(z)\, dz = 2i\int_{r}^{R} \frac{\sin x}{x} dx$

(We used the fact $\frac{\cos x}{x}$ is odd and $\frac{\sin x}{x}$ is even).

We see $\int_{L_1} f(z)\, dz + \int_{L_2} f(z)\, dz \longrightarrow 2i\int_{0}^{+\infty} \frac{\sin x}{x} dx$ as $r \to 0$ and $R \to +\infty$.

Now we deal with $\int_{C_r} f(z)\, dz = \int_{C_r} \frac{e^{iz}}{z} dz$

since $f(z)$ has a simple pole at $z=0$, it has Laurant expansion

$$f(z) = \frac{1}{z} + g(z) \quad (0 < |z| < \infty)$$

where $g(z)$ is a Taylor series $g(z) = \sum_{n=0}^{\infty} a_n z^n$

Since $g(z)$ converges everywhere, in particular it's analytic at $z=0$, so it's analytic on $|z| < \varepsilon$ for some $\varepsilon > 0$. Then it's continuous on $|z| \leq \frac{\varepsilon}{2}$, which is closed and bounded, so $\exists\, M>0$ such that $|g(z)| \leq M \quad \forall\ |z| < \frac{\varepsilon}{2}$.

Now for any $0 < r < \frac{\varepsilon}{2}$,

$$\left| \int_{C_r} g(z)\, dz \right| \leq M \cdot \pi r$$

so $\lim\limits_{r \to 0} \int_{C_r} g(z)\, dz = 0$.

$$\lim_{r\to 0} \int_{C_r} f(z)\, dz = \lim_{r \to 0} \int_{C_r} \frac{1}{z}\, dz = \lim_{r\to 0} \int_{\pi}^{2\pi} \frac{1}{re^{i\theta}} (re^{i\theta})'\, d\theta$$

$$= \lim_{r \to 0} \int_{\pi}^{2\pi} i\, d\theta$$

$$= \lim_{r\to 0} \pi i$$

$$= \pi i$$

Now we conclude: as $r \to 0$ and $R \to +\infty$, by $(*)$

$$2i \int_0^{+\infty} \frac{\sin x}{x}\, dx + \pi i = 2\pi i$$

$$\Rightarrow \quad \int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

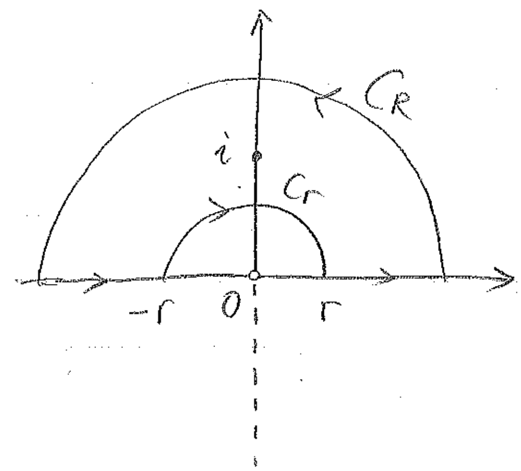
Example. $\displaystyle\int_0^{\infty} \frac{x^a}{(x^2+1)^2}\, dx = \frac{(1-a)\pi}{4\cos(\frac{a\pi}{2})} \quad (-1 < a < 3)$

Let $f(z) = \dfrac{z^a}{(z^2+1)^2} = \dfrac{e^{a \log z}}{(z^2+1)^2} \quad \left(-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$

Take $0<r<1<R$. C_R is the semicircle $z(\theta)=Re^{i\theta}$, $0\le\theta\le\pi$

C_r is the semicircle $z(\theta)=re^{i(\pi-\theta)}$, $0\le\theta\le\pi$

L_1 is $[r,R]$, L_2 is $[-R,-r]$



Observe the pole of $f(z)$ is $z=i$
(Note $z=-i$ is on the branch cut)

We thus get

$$\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz+\int_{L_2} f(z)\,dz+\int_{C_r} f(z)\,dz=2\pi i\,\operatorname*{Res}_{z=i} f(z) \qquad (*)$$

$$f(z)=\frac{e^{a\log z}}{(z^2+1)^2}=\frac{\frac{e^{a\log z}}{(z+i)^2}}{(z-i)^2}$$

so $$\operatorname*{Res}_{z=i} f(z)=\left(\frac{e^{a\log z}}{(z+i)^2}\right)'\Bigg|_{z=i}=-ie^{i\frac{a\pi}{2}}\cdot\frac{1-a}{4}$$

$$\int_{L_1} f(z)\,dz=\int_r^R \frac{e^{a(\ln x)}}{(x^2+1)^2}\,dx=\int_r^R\frac{x^a}{(x^2+1)^2}\,dx$$

$$\int_{L_2} f(z)\,dz=\int_{-R}^{-r}\frac{e^{a(\ln(-x)+i\pi)}}{(x^2+1)^2}\,dx=\int_r^R\frac{e^{a(\ln x)}\cdot e^{a\pi i}}{(x^2+1)^2}\,dx$$

$$=e^{ia\pi}\int_r^R\frac{x^a}{(x^2+1)^2}\,dx$$

For any z on C_r,

$$|f(z)| = \frac{|z|^a}{|z^2+1|^2} \le \frac{|z|^a}{(1-|z|^2)^2} = \frac{r^a}{(1-r^2)^2}$$

$$\left|\int_{C_r} f(z)\,dz\right| \le \frac{r^a}{(1-r^2)^2}\cdot \pi r$$

since $a>-1$, $a+1>0$.

So $$\lim_{r\to 0} \frac{r^{a+1}}{(1-r^2)^2} = \frac{\lim_{r\to 0} r^{a+1}}{\lim_{r\to 0}(1-r^2)^2} = \frac{0}{1} = 0$$

We get $\displaystyle\lim_{r\to 0}\int_{C_r} f(z)\,dz = 0$

For any z on C_R.

$$|f(z)| = \frac{|z|^a}{|z^2+1|^2} \le \frac{|z|^a}{(|z|^2-1)^2} = \frac{R^a}{(R^2-1)^2}$$

Since $a<3$, $a+1<4$

$$\lim_{R\to\infty}\frac{R^a}{(R^2-1)^2} = \lim_{r\to 0}\frac{R^{a+1}}{R^4-2R^2+1} = 0$$

so $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$

Now letting $r\to 0$ and $R\to\infty$, $(*)$ becomes

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx + e^{ia\pi}\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot e^{i\frac{a\pi}{2}}$$

Take the real part of the equation, we get

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx + \cos a\pi\cdot\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot\cos\frac{a\pi}{2}$$

This implies

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)\cos\frac{a\pi}{2}}{2(1+\cos a\pi)} = \frac{\pi(1-a)\cos\frac{a\pi}{2}}{2\cdot 2\cos^2\frac{a\pi}{2}} = \frac{(1-a)\pi}{4\cos\frac{a\pi}{2}}$$

Remark. If we use the imaginary part of the equation, we get

$$\sin a\pi \cdot \int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot \sin\frac{a\pi}{2}$$

which implies

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot\frac{\sin\frac{a\pi}{2}}{\sin a\pi} = \frac{\pi(1-a)}{2}\cdot\frac{1}{2\cos\frac{a\pi}{2}}$$

$$= \frac{(1-a)\pi}{4\cos\frac{a\pi}{2}}$$

as well.