Lemma. (Jordan's Lemma)

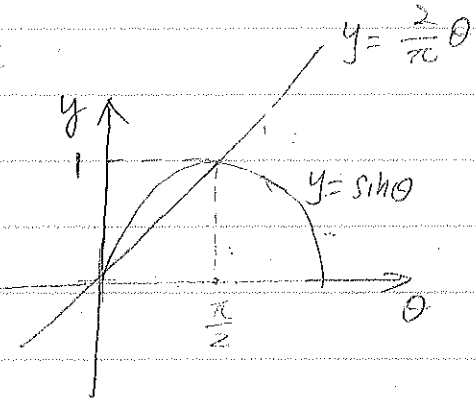
$f(z)$ is analytic at all points in the upper half plane $y \geq 0$ exterior to a circle $|z| = R_0$, and C_R is the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, $(R > R_0)$. If $|f(z)| < M_R$ $\forall z \in C_R$, and $\lim_{R \to +\infty} M_R = 0$, Then:

$$\lim_{R \to +\infty} \int_{C_R} f(z) e^{iaz} \, dz = 0 \qquad \forall\, a > 0.$$

Proof. Since $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$

$R > 0$, then $R \sin\theta \geq R\,\frac{2\theta}{\pi}$

$$e^{-R\sin\theta} \leq e^{-\frac{2R}{\pi}\theta} \quad \text{for } 0 \leq \theta \leq \tfrac{\pi}{2}.$$



This implies

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta} d\theta = -\frac{\pi}{2R} e^{-\frac{2R}{\pi}\theta}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2R}\left(e^{-R} - 1\right)$$

$$= \frac{\pi}{2R}\left(1 - e^{-R}\right)$$

$$< \frac{\pi}{2R}$$

We get the Jordan's Inequality:

$$\int_0^{\pi} e^{-R\sin\theta} d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} < \frac{\pi}{R} \quad \text{for } R > 0$$

$$\int_{C_R} f(z) e^{iaz} dz = \int_0^{\pi} f(Re^{i\theta}) e^{iaRe^{i\theta}} \cdot Rie^{i\theta} d\theta$$

Note that $|f(Re^{i\theta})| \leq M_R$,

$$|e^{iaRe^{i\theta}}| = |e^{iaR(\cos\theta + i\sin\theta)}| = e^{-aR\sin\theta}$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq M_R \cdot R \int_0^{\pi} e^{-aR\sin\theta}\,d\theta < M_R \cdot R \cdot \frac{\pi}{aR} = \frac{\pi}{a} M_R$$

Since $\lim\limits_{R\to+\infty} M_R = 0$, we conclude $\lim\limits_{R\to+\infty} \int_{C_R} f(z)\,dz = 0$

This Lemma has applications in some improper integrals involving sine and cosine functions.

Example $\displaystyle\int_0^{\infty} \frac{x \sin 2x}{x^2+3}\,dx$

Let $f(z) = \dfrac{z}{z^2+3} \cdot e^{i2z} = \dfrac{z}{(z+\sqrt{3}i)(z-\sqrt{3}i)} \cdot e^{i\cdot 2z}$

The singular points of $f(z)$ are $z = \pm\sqrt{3}\,i$.

$$\operatorname*{Res}_{z=\sqrt{3}i}(f) = \frac{\sqrt{3}i}{(\sqrt{3}i+\sqrt{3}i)} \cdot e^{i2\sqrt{3}i} = \frac{1}{2e^{2\sqrt{3}}}$$

Let $R > \sqrt{3}$, C_R be $z(\theta) = \sqrt{3}e^{i\theta}$, $0 \leq \theta \leq \pi$.
$L_R = [-R, R]$, pointing to the right.

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=\sqrt{3}i}(f) = \frac{\pi i}{e^{2\sqrt{3}}}$$

For each $R > \sqrt{3}$, $\left|\dfrac{z}{z^2+3}\right| \leq \dfrac{R}{R^2-3}$ on C_R,

and $\lim\limits_{R\to\infty} \dfrac{R}{R^2-3} = 0$.

By Jordan's Lemma.

$$\lim_{R\to\infty} \int_{C_R} \frac{z}{z^2+3} \cdot e^{i\cdot 2z}\, dz = 0$$

so $$\lim_{R\to\infty} \int_{L_R} f(z)\, dz = \frac{\pi i}{e^{2\sqrt{3}}}$$

$$P.V. \int_{-\infty}^{+\infty} \frac{x\cos 2x}{x^2+3} dx + i\, P.V. \int_{-\infty}^{+\infty} \frac{x \sin 2x}{x^2+3}\, dx = \frac{\pi}{e^{2\sqrt{3}}} i$$

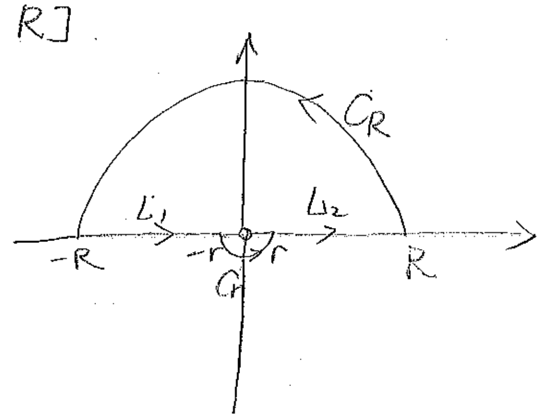
We see $$P.V. \int_{-\infty}^{+\infty} \frac{x\sin 2x}{x^2+3} dx = \frac{\pi}{e^{2\sqrt{3}}}$$

so $$\int_0^{+\infty} \frac{x \sin 2x}{x^2+3} dx = \frac{1}{2}\cdot \frac{\pi}{e^{2\sqrt{3}}} = \frac{\pi}{2e^{2\sqrt{3}}}$$

Example. Evaluate the Dirichlet's Integral. $\int_0^\infty \frac{\sin x}{x} dx$

Let $f(z) = \frac{e^{iz}}{z}$, note $f(z)$ has a simple pole at $z=0$.

Let $0 < r < R$, C_R be the semicircle $z = Re^{i\theta}$, $0 \le \theta \le \pi$.
C_r be the semicircle $z = re^{i\theta}$, $\pi \le \theta \le 2\pi$
L_1 be $[-R, -r]$, L_2 be $[r, R]$.



$$(*) \int_{C_r} f(z)dz + \int_{L_2} f(z)dz + \int_{C_R} f(z)dz + \int_{L_1} f(z)dz$$

$$= 2\pi i \cdot \operatorname{Res}_{z=0} f(z)$$

$$= 2\pi i$$

By Jordan's Lemma, if $R \to \infty$

$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0$. since on C_R, $|\frac{1}{z}| = \frac{1}{R} \to 0$

$$\int_{L_1} f(z)\,dz = \int_{-R}^{-r} \frac{e^{ix}}{x} dx = \int_{-R}^{-r} \frac{\cos x}{x} dx + i\int_{-R}^{-r} \frac{\sin x}{x} dx$$

$$\int_{L_2} f(z)\,dz = \int_{r}^{R} \frac{e^{ix}}{x} dx = \int_{r}^{R} \frac{\cos x}{x} dx + i\int_{r}^{R} \frac{\sin x}{x} dx$$

So $\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2i\int_r^R \frac{\sin x}{x} dx$

(We used the fact $\frac{\cos x}{x}$ is odd and $\frac{\sin x}{x}$ is even).

We see $\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz \longrightarrow 2i\int_0^{+\infty} \frac{\sin x}{x} dx$ as $r \to 0$ and $R \to +\infty$.

Now we deal with $\int_{C_r} f(z)\,dz = \int_{C_r} \frac{e^{iz}}{z} dz$

since $f(z)$ has a simple pole at $z=0$, it has Laurant expansion

$$f(z) = \frac{1}{z} + g(z) \quad (0 < |z| < \infty)$$

where $g(z)$ is a Taylor series $g(z) = \sum_{n=0}^{\infty} a_n z^n$
Since $g(z)$ converges everywhere, in particular it's analytic at $z=0$, so it's analytic on $|z| < \varepsilon$ for some $\varepsilon > 0$. Then it's continuous on $|z| \leq \frac{\varepsilon}{2}$, which is closed and bounded,

so $\exists\, M>0$ such that $|g(z)| \leq M \ \forall\ |z| < \frac{\varepsilon}{2}$.

Now for any $0 < r < \frac{\varepsilon}{2}$,

$$\left| \int_{C_r} g(z)\,dz \right| \leq M \cdot \pi r$$

so $\lim\limits_{r\to 0} \int_{C_r} g(z)\,dz = 0$.

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = \lim_{r\to 0} \int_{C_r} \frac{1}{z}\,dz = \lim_{r\to 0} \int_{\pi}^{2\pi} \frac{1}{re^{i\theta}} (re^{i\theta})'\,d\theta$$

$$= \lim_{r\to 0} \int_{\pi}^{2\pi} i\,d\theta$$

$$= \lim_{r\to 0} \pi i$$

$$= \pi i$$

Now we conclude: as $r \to 0$ and $R \to +\infty$, by $(*)$

$$2i \int_0^{+\infty} \frac{\sin x}{x}\,dx + \pi i = 2\pi i$$

$$\Rightarrow \quad \int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

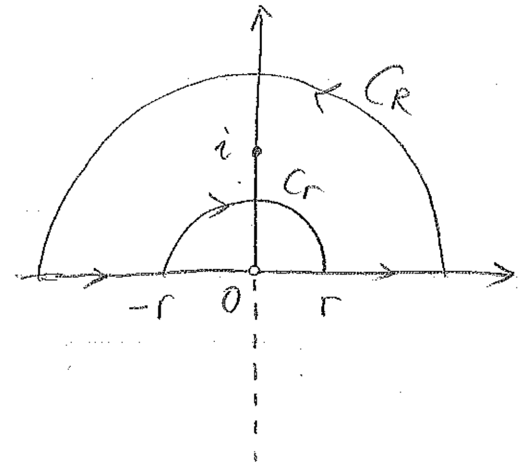
Example. $\displaystyle\int_0^{\infty} \frac{x^a}{(x^2+1)^2}\,dx = \frac{(1-a)\pi}{4\cos(\frac{a\pi}{2})} \quad (-1 < a < 3)$

Let $f(z) = \dfrac{z^a}{(z^2+1)^2} = \dfrac{e^{a\log z}}{(z^2+1)^2} \quad \left(-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$

Take $0<r<1<R$. C_R is the semicircle $z(\theta)=Re^{i\theta}$, $0\le\theta\le\pi$

C_r is the semicircle $z(\theta)=re^{i(\pi-\theta)}$, $0\le\theta\le\pi$

L_1 is $[r, R]$, L_2 is $[-R, -r]$



Observe the pole of $f(z)$ is $z=i$

(Note $z=-i$ is on the branch cut)

We thus get

$$\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz+\int_{L_2} f(z)\,dz+\int_{C_r} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) \qquad (*)$$

$$f(z)=\frac{e^{a\log z}}{(z^2+1)^2}=\frac{\frac{e^{a\log z}}{(z+i)^2}}{(z-i)^2}$$

so $$\operatorname*{Res}_{z=i} f(z)=\left(\frac{e^{a\log z}}{(z+i)^2}\right)'\Bigg|_{z=i} = -ie^{i\frac{a\pi}{2}}\cdot\frac{1-a}{4}$$

$$\int_{L_1} f(z)\,dz=\int_r^R \frac{e^{a(\ln x)}}{(x^2+1)^2}\,dx=\int_r^R\frac{x^a}{(x^2+1)^2}\,dx$$

$$\int_{L_2} f(z)\,dz=\int_{-R}^{-r}\frac{e^{a(\ln(-x)+i\pi)}}{(x^2+1)^2}\,dx=\int_r^R\frac{e^{a(\ln x)}\cdot e^{a\pi i}}{(x^2+1)^2}\,dx$$

$$=e^{ia\pi}\int_r^R\frac{x^a}{(x^2+1)^2}\,dx$$

For any z on C_r,

$$|f(z)| = \frac{|z|^a}{|z^2+1|^2} \le \frac{|z|^a}{(1-|z|^2)^2} = \frac{r^a}{(1-r^2)^2}$$

$$\left|\int_{C_r} f(z)\,dz\right| \le \frac{r^a}{(1-r^2)^2}\cdot \pi r$$

since $a>-1$, $a+1>0$.

So $$\lim_{r\to 0} \frac{r^{a+1}}{(1-r^2)^2} = \frac{\lim_{r\to 0} r^{a+1}}{\lim_{r\to 0}(1-r^2)^2} = \frac{0}{1} = 0$$

We get $\lim_{r\to 0}\int_{C_r} f(z)\,dz = 0$

For any z on C_R.

$$|f(z)| = \frac{|z|^a}{|z^2+1|^2} \le \frac{|z|^a}{(|z|^2-1)^2} = \frac{R^a}{(R^2-1)^2}$$

Since $a<3$, $a+1<4$

$$\lim_{R\to\infty} \frac{R^a}{(R^2-1)^2} = \lim_{r\to 0} \frac{R^{a+1}}{R^4-2R^2+1} = 0$$

so $\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$

Now letting $r\to 0$ and $R\to\infty$, (*) becomes

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx + e^{ia\pi}\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot e^{i\frac{a\pi}{2}}$$

Take the real part of the equation, we get

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx + \cos a\pi\cdot\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot\cos\frac{a\pi}{2}$$

This implies

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)\cos\frac{a\pi}{2}}{2(1+\cos a\pi)} = \frac{\pi(1-a)\cos\frac{a\pi}{2}}{2\cdot 2\cos^2\frac{a\pi}{2}} = \frac{(1-a)\pi}{4\cos\frac{a\pi}{2}}$$

Remark. If we use the imaginary part of the equation, we get

$$\sin a\pi \cdot \int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot \sin\frac{a\pi}{2}$$

which implies

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)}{2}\cdot\frac{\sin\frac{a\pi}{2}}{\sin a\pi} = \frac{\pi(1-a)}{2}\cdot\frac{1}{2\cos\frac{a\pi}{2}}$$

$$= \frac{(1-a)\pi}{4\cos\frac{a\pi}{2}}$$

as well.