IMPROPER INTEGRALS

Definition. $f : \mathbb{R} \to \mathbb{R}$ is a real function. The improper integral $\int_{-\infty}^{+\infty} f(x)\,dx$ is defined to be the number

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R_1 \to -\infty} \int_{R_1}^{0} f(x)\,dx + \lim_{R_2 \to +\infty} \int_{0}^{R_2} f(x)\,dx$$

if both limits converge.

Definition. $f : \mathbb{R} \to \mathbb{R}$ is a real function. The Cauchy Principal Value of the improper integral is defined to be

$$\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R \to +\infty} \int_{-R}^{R} f(x)\,dx$$

if the limit converges.

Remark Note that the existence of $\int_{-\infty}^{+\infty} f(x)\,dx$ implies the convergence of $\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$, but the reverse is not always true.

For example $\int_{-\infty}^{+\infty} x\,dx$ diverges, but $\text{P.V.} \int_{-\infty}^{+\infty} x\,dx = 0$

Lemma. If $f : \mathbb{R} \to \mathbb{R}$ is an even function, and $\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$ converges, then $\int_{-\infty}^{+\infty} f(x)\,dx$ converges and

$$\int_{-\infty}^{+\infty} f(x)\,dx = \text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$$

Moreover,

$$\int_{-\infty}^{0} f(x)\,dx = \int_{0}^{+\infty} f(x)\,dx = \frac{1}{2}\,\text{P.V.} \int_{-\infty}^{+\infty} f(x)\,dx$$

A useful application of residues is to compute improper integrals.

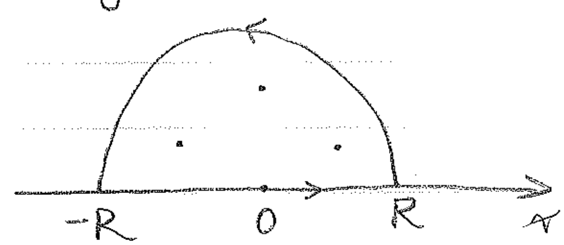
Example. $\displaystyle\int_0^{\infty} \frac{1}{x^6+1}\,dx$

This is an even function, so we can compute $\displaystyle \text{P.V.}\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx$ first.

Consider the complex function $f(z) = \dfrac{1}{z^6+1}$

The singular points of $f(z)$ are the zeros of z^6+1, which are $e^{\frac{\pi}{6}i}$, $e^{\frac{\pi}{2}i}$, $e^{\frac{5\pi}{6}i}$, $e^{\frac{7}{6}\pi i}$, $e^{\frac{3}{2}\pi i}$, $e^{\frac{11}{6}\pi i}$.

Construct a semicircle with centre at 0, radius $R>1$, C_R given by $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Let L_R be the line segment $[-R, R]$, pointing to right.



Then:

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = \Big[\operatorname*{Res}_{z=e^{\frac{\pi}{6}i}}(f) + \operatorname*{Res}_{z=e^{\frac{\pi}{2}i}}(f) + \operatorname*{Res}_{z=e^{\frac{5}{6}\pi i}}(f)\Big]\cdot 2\pi i$$

Note $e^{\frac{\pi}{6}i}$, $e^{\frac{\pi}{2}i}$, $e^{\frac{5}{6}\pi i}$ are zeros of order 1 for z^6+1,

So $\displaystyle\operatorname*{Res}_{z=e^{\frac{\pi}{6}i}}(f) = \frac{1}{6(e^{\frac{\pi}{6}i})^5} = -\frac{1}{6}e^{\frac{\pi}{6}i}$

$\displaystyle\operatorname*{Res}_{z=e^{\frac{\pi}{2}i}}(f) = \frac{1}{6(e^{\frac{\pi}{2}i})^5} = -\frac{1}{6}e^{\frac{\pi}{2}i}$

$\displaystyle\operatorname*{Res}_{z=e^{\frac{5}{6}\pi i}}(f) = \frac{1}{6(e^{\frac{5}{6}\pi i})^5} = -\frac{1}{6}e^{\frac{5}{6}\pi i}$

We get

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = -\frac{1}{6}\left(e^{\frac{\pi}{6}i} + e^{\frac{\pi}{2}i} + e^{\frac{5}{6}\pi i}\right)\cdot 2\pi i$$

$$= -\frac{1}{6}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i + i + \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right)\cdot 2\pi i$$

$$= -\frac{1}{6} \times 2i \times 2\pi i$$

$$= \frac{2}{3}\pi$$

Note $\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{R^6 - 1}\cdot \pi R \longrightarrow 0$ as $R \to +\infty$.

We get $\lim_{R\to+\infty} \int_{C_R} f(z)\,dz = 0$

So $P.V. \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to+\infty} \int_{L_R} f(z)\,dz = \frac{2}{3}\pi$

$$\int_0^{+\infty} f(x)\,dx = \frac{1}{2} \times \frac{2}{3}\pi = \frac{\pi}{3}$$

Example $\int_0^{+\infty} \frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{5\pi}{16e^4}$:

Let $f(z) = \frac{e^{i2z}}{(z^2+4)^2}$

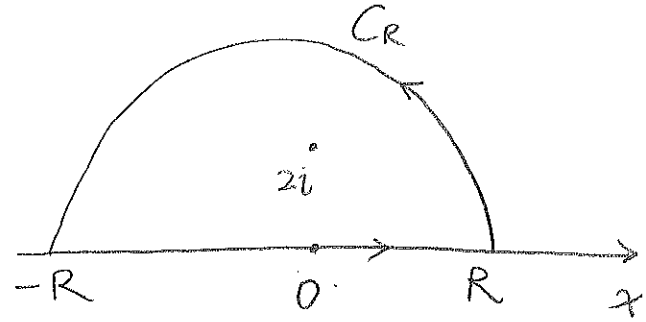
The singular points of $f(z)$ are zeros of $(z^2+4)^2$: $2i$ & $-2i$

Let $R > 2$, C_R the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$.
L_R the line segment $[-R, R]$, pointing to the right

$$\int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=2i}(f)$$

$$f(z) = \frac{\frac{e^{i\cdot 2z}}{(z+2i)^2}}{(z-2i)^2}.$$

Observe that $\phi(z) = \frac{e^{i\cdot 2z}}{(z+2i)^2}$ is analytic at $z=2i$ and $\phi(2i) \neq 0$.



$$\phi'(z) = \frac{2ie^{i2z}(z+2i)^2 - 2e^{i\cdot 2z}(z+2i)}{(z+2i)^4}$$

$$\phi'(2i) = \frac{2i\cdot e^{-4}\cdot(4i)^2 - 2e^{-4}(4i)}{(4i)^4}$$

$$= \frac{-2\times 4^2 i - 8i}{4^4 e^4}$$

$$= \frac{-5i}{32e^4}$$

So $\operatorname*{Res}_{z=2i}(f) = \phi'(2i) = \frac{-5i}{32e^4}$

$$\int_{L_R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \cdot \frac{-5i}{32e^4} = \frac{5\pi}{16e^4}$$

If z is on C_R, we see

$|z^2+4| \geq |z^2| - 4 = |z|^2 - 4 = R^2 - 4.$

$|e^{i2z}| = |e^{i2(x+iy)}| = e^{-2y} \leq 1$ since $y \geq 0$ on C_R.

So $\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{(R^2-4)^2}\cdot \pi R \longrightarrow 0$ as $R \to +\infty$

we get $\lim_{R\to+\infty} \int_{L_R} f(z)\,dz = \frac{5\pi}{16e^4}$, i.e.

$$\text{P.V.} \int_{-\infty}^{+\infty} \frac{\cos 2z}{(z^2+4)^2}dx + i\cdot \text{P.V} \int_{-\infty}^{+\infty} \frac{\sin 2z}{(z^2+4)^2}dx = \frac{5\pi}{16e^4}.$$