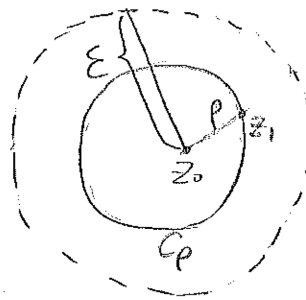
Lemma. Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighbourhood of z_0: $|z-z_0|<\varepsilon$ in which f is analytic. Then $f(z) \equiv f(z_0)$ on this neighbourhood.

Proof For any z_1 in the neighbourhood $|z-z_0|<\varepsilon$.

let $\rho = |z_1 - z_0| < \varepsilon$, C_ρ is the circle $z(t) = z_0 + \rho e^{it}$, $0 \leq t \leq 2\pi$.

By assumption, f is analytic on and inside C_ρ, so by Cauchy Integral Formula,



$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{it})}{\rho e^{it}} \cdot i \cdot \rho e^{it}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{it})\,dt$$

so $$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\rho e^{it})\,dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$$

$$\leq \frac{1}{2\pi}\cdot |f(z_0)| \cdot 2\pi$$

$$= |f(z_0)|$$

we get $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)| - |f(z_0+\rho e^{it})|\,dt = |f(z_0)| - \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt = 0.$$

but $|f(z_0)| - |f(z_0+\rho e^{it})| \geq 0$ for all $t \in [0, 2\pi]$

we conclude

$$|f(z_0)| = |f(z_0+\rho e^{it})| \quad \forall\ t \in [0, 2\pi]$$

Since the above holds for any $\rho < \varepsilon$, we conclude

$|f(z_0)| = |f(z)|$ for any $|z - z_0| < \varepsilon$.

$\Rightarrow$ $|f(z)|$ is constant on the domain $|z - z_0| < \varepsilon$.

Recall we've proved long ago in an example that this implies $f(z) \equiv C$ on the domain.

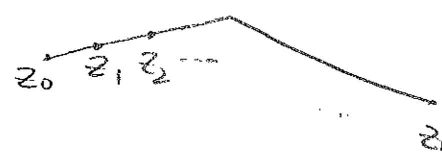
So we finish the proof.

Theorem. (Maximum Modulus Principle) If a function f is analytic and not constant in a domain D, then $|f(z)|$ doesn't have a maximum in D.

Proof. Suppose $f(z)$ takes maximum at $z_0 \in D$, i.e. $|f(z)| \le |f(z_0)|$ for any $z \in D$.

Now for any $z \in D$, we can connect z_0 to z by a polygonal path L.

Let d be the shortest distance between points on L and boundary of D



(In case $D = \mathbb{C}$, we can take any $d > 0$)

Then L is covered by finitely many open balls of radius d, with centre at $z_0, z_1, \ldots, z_n = z$, respectively, such that $|z_i - z_{i-1}| < d$ for all $i = 1, 2, \ldots, n$.

By the Lemma, f is constant on $B(z_0, d)$, and $z_1 \in B(z_0, d)$ so $f(z_1) = f(z_0) \geq f(z)$ for any $z \in D$. Apply the Lemma to $B(z_1, d)$

we get f is constant on $B_1(z_0, d)$, and we continue this process n times to conclude $f(z) = f(z_n) = f(z_{n-1}) = \cdots = f(z_0)$. Since $z \in D$ is arbitrary at beginning, we conclude f is constant

Corollary. If f is continuous on a bounded closed region R and f is analytic and not constant in the interior of R. Then $\max_{z\in R}|f(z)|$ is obtained only at some points on the boundary of R

Proof. Since f is continuous and R is closed and bounded, $\max_{z\in R}|f(z)|$ exists. If it appears in the interior, then by the Maximum Modulus Principle, f is constant in the interior of R, contradiction.

Remark. In the proof of the Lemma, the formula

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})\, dt$$

is called the Gauss' Mean Value Theorem. It tells us if f is analytic on and inside a circle $|z-z_0| = \rho$, then $f(z_0)$ is the arithmetic mean of the values on the circle.