CAUCHY INTEGRAL FORMULA

The Cauchy Integral Formula is a fundamental result in complex analysis, which tells us how the value of an analytic function at $z_0 \in \mathbb{C}$ is related to integral around it.

Theorem. (Cauchy Integral Formula)
Let f be analytic everywhere inside and on a simple closed contour C, taken counterclockwise direction. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Proof. Take a circle C_r of small radius $r>0$, centered at z_0, counterclockwise oriented and C_r is in the interior of C.

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = \int_{C_r}\frac{f(z)}{z-z_0}\,dz - \int_{C_r}\frac{f(z_0)}{z-z_0}\,dz$$

$$= \int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz$$

f is analytic at z_0, so it's continuous at z_0. For any $\varepsilon>0$, $\exists\ \delta>0$ such that $|z-z_0|<\delta \Rightarrow |f(z)-f(z_0)|<\varepsilon$.

Now take $\delta' = \min\{r, \delta\}$, then

$$\left|\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| = \left|\int_{C_{\delta'}}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\delta'}\cdot 2\pi\delta' = 2\pi\varepsilon$$

Since $\varepsilon>0$ is arbitrary, we conclude

$$\int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = 0,$$

i.e. $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Example. C is the positively oriented circle $|z|=1$.

$$\int_C \frac{\cos z}{z(z^2+9)}\,dz = \int_C \frac{\frac{\cos z}{z^2+9}}{z-0}\,dz = 2\pi i \cdot \frac{\cos 0}{0^2+9} = \frac{2\pi i}{9}$$

Theorem. f is analytic inside and on a simple closed contour C, which is positively oriented. If z_0 is in the interior of C, then for each $n \in \mathbb{N}$, $f^{(n)}(z_0)$ exists and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Proof. We will show the case for $n=1$. The general case can be proved by induction using similar idea, but more technical details will be involved. The reader may refer to Ahlfors' Complex Analysis book for more details.

We know $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z}\left[\int_C \frac{f(z)}{z-z_0-\Delta z}\,dz - \int_C \frac{f(z)}{z-z_0}\,dz\right]$$

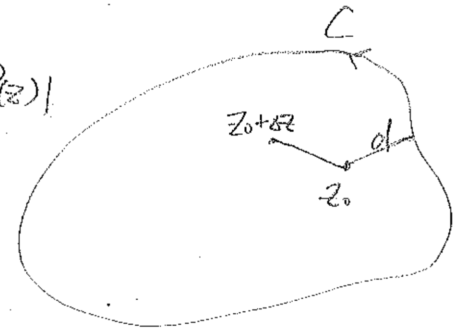
$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{\Delta z}\cdot\left(\frac{1}{z-z_0-\Delta z} - \frac{1}{z-z_0}\right)dz$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz$$

$$f'(z_0) - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz = \lim_{\Delta z \to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} - \frac{f(z)}{(z-z_0)^2}\,dz$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i}\int_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz$$

Let $d = \min_{z \in C} |z - z_0|$ and L is the arclength of the contour C, $M = \max_{z \in C} |f(z)|$.



We may only consider the case $|\Delta z| < d$, since we'll take the limit as $\Delta z \to 0$.

so
$$\left| \int_C \frac{f(z)\,\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz \right| \le \frac{|\Delta z| \cdot M}{(d - |\Delta z|)\, d^2} \cdot L,$$

$\left(\text{since } |z - z_0 - \Delta z| \ge \big|\, |z - z_0| - |\Delta z| \,\big| \ge d - |\Delta z| > 0 \right)$

We now see as $\Delta z \to 0$, $\int_C \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\, dz \to 0$

so $f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz$

Example. Compute $\int_C \frac{e^z}{z^4}\, dz$, where C is the unit circle $|z| = 1$ positively oriented.

$$\int_C \frac{e^z}{z^4}\, dz = \frac{2\pi i}{3!} \cdot \frac{3!}{2\pi i} \int_C \frac{e^z}{(z - 0)^{3+1}}\, dz = \frac{2\pi i}{3!} \cdot \left. \frac{d^3 e^z}{dz^3} \right|_{z=0} = \frac{\pi}{3} i$$

Remark. We sometimes want to express the function $f^{(n)}(z)$ by the Cauchy Integral Formula, so we need to change the variables in the previous case:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s - z)^{n+1}}\, ds$$

Theorem. If f is analytic at z_0, then f' is analytic at z_0.

Proof. If f is analytic at z_0, by definition, there's a open neighbourhood U of z_0 on which f is analytic. So we can take a small circle C centred at z_0 such that $C \subseteq U$. Then by Cauchy Integral Formula, for each z inside the circle, $f''(z)$ exists and

$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds$$

so f' is analytic at z_0.

Corollary. If f is analytic at z_0, then for each $n \in \mathbb{N}$, $f^{(n)}$ is analytic at z_0

Corollary. If $f(z) = u(x,y) + iv(x,y)$ is analytic at $z_0 = (x_0, y_0)$, then all the partial derivatives of any order of u and v exist at (x_0, y_0)

Theorem. f is continuous on a domain D. If $\int_C f(z)\,dz = 0$ for every closed contour inside D, then f is analytic on D

Proof. $\int_C f(z)\,dz = 0$ for every closed curve on $D \Rightarrow f$ has an antiderivative F on D, i.e. $F'(z) = f(z)$.

Since f is the derivative of F, $f = F'$ is analytic at every point in D, so f is analytic on D.

Theorem. (Cauchy's Inequality) f is analytic inside and on a positively oriented circle C_R centred at z_0 with radius R. $M_R = \max_{z \in C} |f(z)|$, then: $|f^{(n)}(z_0)| \leq \frac{n!\,M_R}{R^n}$

Proof. By Cauchy Integral Formula,

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R$$

$$= \frac{n!\, M_R}{R^n}$$

Theorem. (Liouville's Theorem) If f is an entire function that is bounded, then $f(z) \equiv c$ for some $c \in \mathbb{C}$

Proof. Since f is bounded, there's $M > 0$ such that $|f(z)| < M$ for all $z \in \mathbb{C}$.

f is entire, so for any $z \in \mathbb{C}$, we take the circle C_R centred at z with radius R, and apply the previous Theorem:

$$|f'(z)| \leq \frac{M_R}{R} < \frac{M}{R}$$

Since this is true for any $R > 0$, letting $R \to +\infty$ we get $|f'(z)| = 0$. i.e $f'(z) = 0$ $\forall z \in \mathbb{C}$

We conclude f is a constant function

Theorem. (Fundamental Theorem of Algebra)
Any non constant polynomial $p(z) \in \mathbb{C}[z]$ has at least one root in $\mathbb{C}$. i.e $\exists z_0 \in \mathbb{C}$ such that $p(z_0) = 0$

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a nonconstant polynomial. Suppose $p(z)$ has no root, then $\frac{1}{p(z)}$ is an entire function.

If we can show $\frac{1}{p(z)}$ bounded, then by Liouville's Theorem, $\frac{1}{p(z)} \equiv C \Rightarrow p(z) \equiv \frac{1}{C}$. Contradicts to the assumption, we'll finish the proof.

let $w = \frac{p(z)}{z^n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}$

so $|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|}$

We can take a large $R>0$ such that whenever $|z| \geq R$,

$$\frac{|a_0|}{|z|^n} < \frac{|a_n|}{2n}, \quad \frac{|a_1|}{|z|^{n-1}} < \frac{|a_n|}{2n}, \quad \ldots, \quad \frac{|a_{n-1}|}{|z|} < \frac{|a_n|}{2n}$$

$$|w| \leq n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}$$

so for $|z| > R$, $|p(z)| = |a_n + w| \cdot |z^n| \geq \left(|a_n| - \frac{|a_n|}{2}\right) \cdot R^n$
$= \frac{|a_n|}{2} \cdot R^n$

we see $\left|\frac{1}{p(z)}\right|$ is bounded for $|z| > R$.

On $|z| \leq R$, $\left|\frac{1}{p(z)}\right|$ is also bounded since $|z| \leq R$ is a bounded and closed region and $\frac{1}{p(z)}$ is continuous.

So we conclude $\frac{1}{p(z)}$ is bounded