Theorem. If f is analytic throughout a simply-connected domain D, then $\int_C f(z)\,dz = 0$ for every closed contour C lying in D

Proof. If C is simple closed in the simply-connected domain D, then C and its interior are all in D, so f is analytic on C and in its interior, by Cauchy-Goursat Theorem $\therefore \int_C f(z)\,dz = 0$

More generally, if C has finitely many self-intersections, we can regard C as a sum of simple closed contours. The result also follows.

If there're infinitely many self-intersection, we need more tricks, we'll not discuss here.

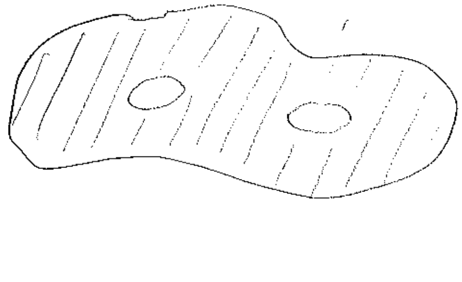
Example. $\int_C \frac{1}{z}\,dz = 0$ for all simple closed contour C that doesn't enclose the origin.

Corollary. If f is an analytic function throughout a simply-connected domain D, then f has an antiderivative on D

Corollary. Entire functions have antiderivatives

Definition. If a domain is not simply-connected, we say it's multiply connected

Example.



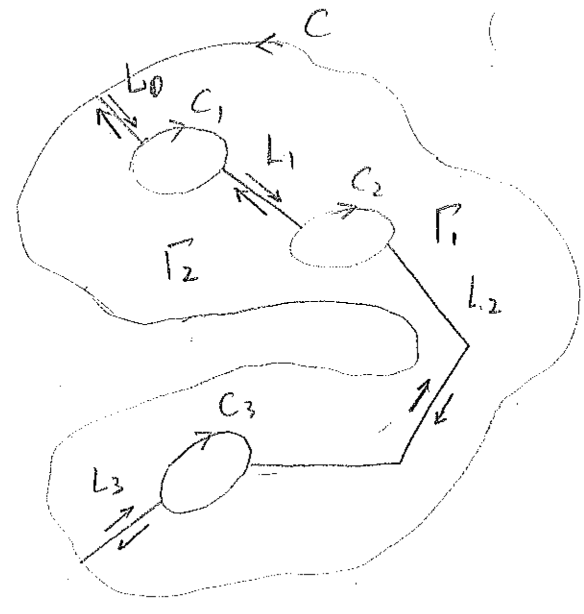
Theorem. If C is a simple closed contour, counterclockwisely oriented, and C_k $(k=1,2,\dots,n)$ are simple closed contours interior to C, all in the clockwise direction, that are disjoint and whose interior have no points in common.

If f is analytic in the multiply connected domain consisting of points inside C and exterior to each C_i, then:

$$\int_C f(z)\,dz + \sum_{i=1}^{n}\int_{C_i} f(z)\,dz = 0$$

Proof. We use polygonal paths to connect C to C_1, C_1 to C_2, ..., C_{n-1} to C_n, and C_n to C, and label these paths by $L_0, L_1, \dots, L_n$.

Then two simple closed paths Γ_1 and Γ_2 are formed.



By Cauchy-Goursat,

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

The integral on L_i cancel with each other, so we get

$$\int_C f(z)\,dz + \sum_{i=1}^{n}\int_{C_i} f(z)\,dz = 0$$

Corollary. (Principle of Deformation of Paths)

Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If f is analytic in the closed region consisting of those contours and all points between them, then:

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Remark. More generally, in the theorem, if we take $C_1, \dots, C_n$ counterclockwisely oriented, and all the other conditions unchanged, then:

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

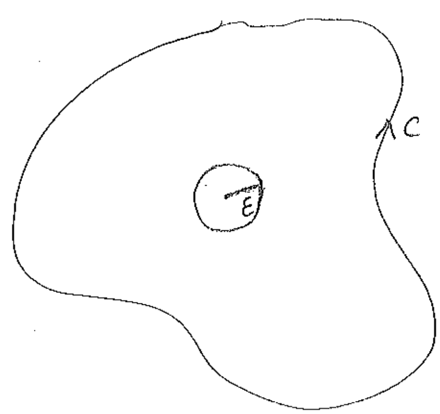
Example. If C is a simple closed contour counterclockwisely oriented, enclosing 0 in its interior, then

claim: $\displaystyle\int_C \frac{1}{z}\,dz = 2\pi i$.

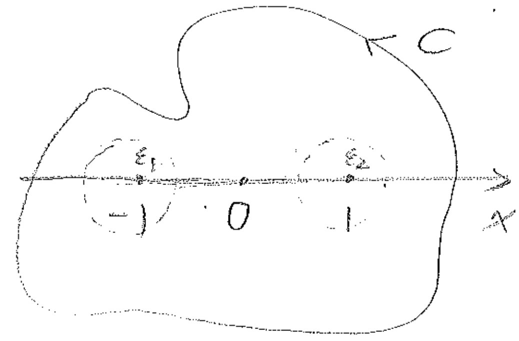
We can take a circle C_ε of small radius $\varepsilon > 0$ centered at 0 such that the circle is in the interior of C.

Then By the Corollary.

$$\int_C \frac{1}{z}\,dz = \int_{C_\varepsilon} \frac{1}{z}\,dz = 2\pi i$$



Example. C is a simple closed contour, counterclockwisely oriented, enclosing 1 and -1 in its interior. Let's try to compute $\int_C \frac{1}{z^2-1}\,dz$.



Take small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the circle

C_1: $z_1(t) = -1 + \varepsilon_1 e^{it}$, $0 \le t \le 2\pi$ and

C_2: $z_2(t) = 1 + \varepsilon_2 e^{it}$, $0 \le t \le 2\pi$ are

in the interior of C and C_1, C_2 disjoint.

Then since $\frac{1}{z^2-1}$ is analytic inside C and exterior of C_1, C_2

$$\int_C \frac{1}{z^2-1}\,dz = \int_{C_1} \frac{1}{z^2-1}\,dz + \int_{C_2} \frac{1}{z^2-1}\,dz$$

$$= \frac{1}{2}\left[\int_{C_1} \frac{1}{z-1}\,dz - \int_{C_1} \frac{1}{z+1}\right]$$

$$+ \frac{1}{2}\left[\int_{C_2} \frac{1}{z-1}\,dz - \int_{C_2} \frac{1}{z+1}\,dz\right]$$

$$= -\frac{1}{2}\int_{C_1} \frac{1}{z+1}\,dz + \frac{1}{2}\int_{C_2} \frac{1}{z-1}\,dz$$

$$= -\frac{1}{2}\cdot\int_0^{2\pi} \frac{1}{\varepsilon_1 e^{it}}\cdot i\varepsilon_1 e^{it}\,dt + \frac{1}{2}\int_0^{2\pi} \frac{1}{\varepsilon_2 e^{it}}\, i\varepsilon_2 e^{it}\,dt$$

$$= -\pi i + \pi i$$

$$= 0$$