CAUCHY-GOURSAT THEOREM

Theorem. (Early Version by Cauchy)
C is a simple closed contour, R is the set of all points interior to or on C. If f is analytic on R, and f' is continuous, then $\int_C f(z)\,dz = 0$.

Proof. We let $f(z) = u(x,y) + iv(x,y)$ and C be $z(t) = x(t) + iy(t)$ $a \le t \le b$.

Then

$$\int_C f(z)\,dz = \int_a^b \left(u(x(t),y(t)) + i\,v(x(t),y(t))\right)\cdot\left(x'(t) + iy'(t)\right) dt$$

$$= \int_a^b u(x(t),y(t))\,x'(t) - v(x(t),y(t))\cdot y'(t)\;dt \;+$$

$$i\int_a^b u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\;dt$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Green's Theorem

$$\stackrel{\text{Green's Theorem}}{=} \left(\iint_R \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\,dA\right) + i\left(\iint_R \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,dA\right)$$

$$= \iint_R -(v_x + u_y)\,dA + i\iint_R (u_x - v_y)\,dA$$

Cauchy-Riemann Equations

$$\stackrel{\text{Cauchy-Riemann Equations}}{=} 0$$

Remark. In this early version, f' is assumed to be continuous because we need to satisfy the condition for Green's Theorem

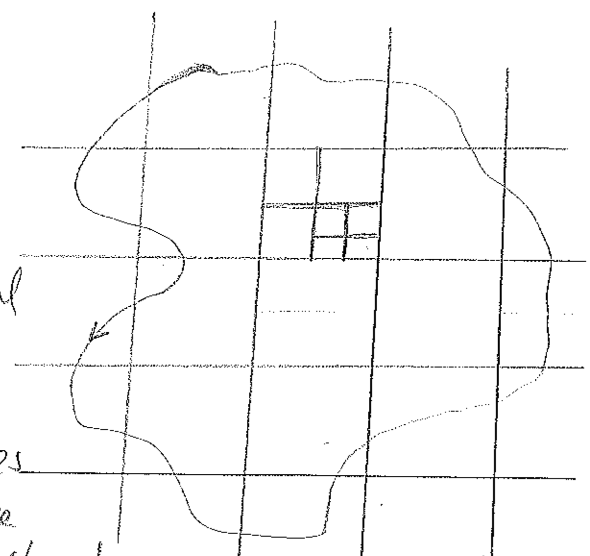
Lemma. Let f be analytic throughout a closed region R consisting of the points interior to a positively oriented simple closed contour C, together with the points on C itself. For any positive number $\varepsilon>0$, the region R can be covered with a finite number of squares and partial squares, indexed by $j=1,2,\dots,n$ such that in each one there is a fixed point z_j for which the inequality

$$\left|\frac{f(z)-f(z_j)}{z-z_j}-f'(z_j)\right|<\varepsilon$$

is satisfied by all points other than z_j in that square or partial square.

Proof. We first construct equally spaced horizontal and vertical lines that separate the region R into squares and partial squares.

If in one of the squares or partial squres, we cannot find a z_j, then we subdivide it into four smaller squares of equal size.



We claim that after finitely times of subdivisions, each small square will satisfy the requirements of the lemma.

We'll prove by contradiction. Suppose σ_k, $k\in\mathbb{N}$ is a nested sequence of squares, the diameter of σ_k is $\frac{b}{2^k}$, where b is a constant. It can be shown that there exits a $z_0\in\bigcap_{k=1}^{\infty}\sigma_k$.

Since f is analytic at z_0, for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$.

but then we see if K is big enough, σ_K has diameter less than ε. then for any $z \in \sigma_K$, $\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$.

Contradict to the assumption.

Theorem (Cauchy - Goursat)

If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

Proof By the previous Lemma, given any $\varepsilon > 0$. we can cover the region R enclosed by C by squares or partial squares σ_j such that $\exists\ z_j \in \sigma_j$. $\forall z \in \sigma_j \setminus \{z_j\}$

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon.$$

Define

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & \text{if } z \in \sigma_j \setminus \{z_j\} \\ 0 & \text{if } z = z_j \end{cases}$$

Then δ_j is a continuous function on σ_j. such that $|\delta_j(z)| < \varepsilon$

We can rewrite the above as

$$f(z) = f(z_j) + (\delta_j(z) + f'(z))(z - z_j)$$

$$= f(z_j) - z_j f'(z_j) + f'(z) z + (z - z_j)\delta_j(z)$$

We thus get: $f(z)$ integrates along C_j, the boundary of σ_j (counterclockwise)

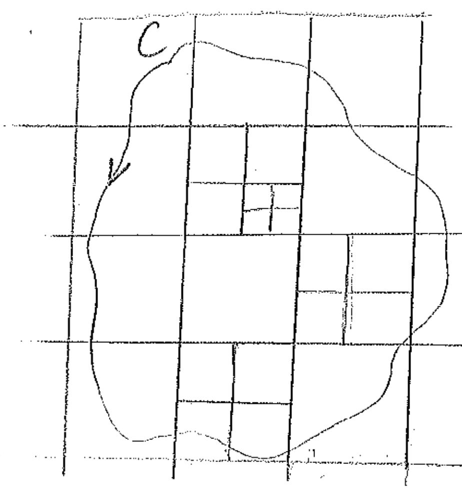
$$\int_{C_j} f(z)\,dz = \int_{C_j} f(z_j) - z_j f'(z_j) + f'(z_j) z + (z - z_j)\delta_j(z)\,dz$$

$$= (f(z_j) - z_j f'(z_j)) \int_{C_j} 1\,dz + f'(z_j)\int_{C_j} z\,dz + \int_{C_j} (z - z_j)\delta_j(z)\,dz$$

$$= \int_{C_j} (z - z_j)\delta_j(z)\,dz$$

Another key observation is that

$$\int_C f(z)\,dz = \sum_j \int_{C_j} f(z)\,dz = \sum_j \int_{C_j} (z - z_j)\delta_j(z)\,dz$$



If C_j is a square, let S_j be the length of an edge of σ_j. A_j is the area of σ_j, then

$$\left|\int_{C_j} (z - z_j)\delta_j(z)\,dz\right| \le \int_{C_j} |z - z_j|\,|\delta_j(z)|\,dz$$

$$\le (\sqrt{2} S_j \cdot \varepsilon)\cdot(4S_j)$$

$$= 4\sqrt{2}\,\varepsilon A_j$$

If C_j is the boundary of a partial square,

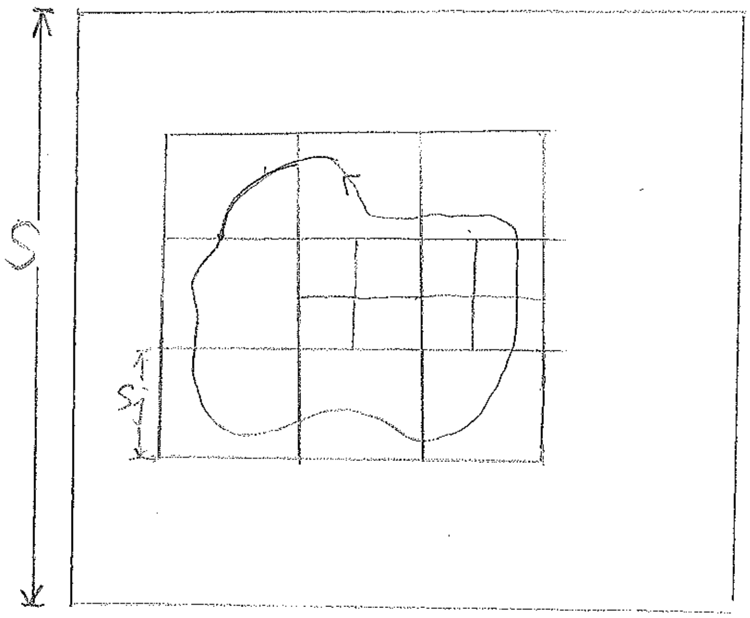
$$\left|\int_{C_j} (z - z_j)\delta_j(z)\,dz\right| \le (\sqrt{2} S_j \varepsilon)\cdot(4S_j + L_j)$$

$$< 4\sqrt{2} A_j \varepsilon + \sqrt{2} S L_j \varepsilon$$

where L_j is the part of C_j which is also part of C.

S is the length of a side of some square that encloses the entire contour C as well as all of the squares covering R. Now let

L be the arc length of C



$$\left|\int_C f(z)\,dz\right| \le \sum_j \left|\int_{C_j} f(z)\,dz\right|$$

$$\le 4\sqrt{2}\,\varepsilon S^2 + \sqrt{2}\,S L \varepsilon$$

$$= (4\sqrt{2}\,S^2 + \sqrt{2}\,SL)\,\varepsilon$$

Since S and L are constants, we see as $\varepsilon \to 0$, we get

$$\int_C f(z)\,dz = 0$$

Definition. A simply connected domain D is a domain such that every simple closed contour within it encloses only points of D.

Example.
- $\mathbb{C}$ is a simply connected domain
- The interior of a simple closed curve is a simply-connected domain.
- The annulus region $1 < |z| < 2$ is NOT a simply-connected domain