Theorem. (Early Version by Cauchy)

C is a simple closed contour. \( R \) is the set of all points interior to or on \( C \). If \( f \) is analytic on \( R \), and \( f' \) is continuous, then \( \int_C f(z) \, dz = 0 \).

Proof. We let \( f(z) = u(x,y) + iv(x,y) \) and \( C \) be \( z(t) = x(t) + iy(t) \) \( a \leq t \leq b \).

Then

\[
\int_C f(z) \, dz = \int_a^b \left( u(x(t), y(t)) + iv(x(t), y(t)) \right) \, (x'(t) + iy'(t)) \, dt
\]

\[
= \int_a^b u(x(t), y(t)) x'(t) \, dt - v(x(t), y(t)) y'(t) \, dt + \int_a^b u(x(t), y(t)) y'(t) \, dt + v(x(t), y(t)) x'(t) \, dt
\]

\[
= \int_C u \, dx - v \, dy + i \int_C v \, dx + u \, dy
\]

Green's Theorem

\[
= \left( \iint_R \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \, dA \right) + i \left( \iint_R \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \, dA \right)
\]

Cauchy-Riemann Equations

\[
= \iint_R - (u_x + u_y) \, dA + i \int_R (u_y - u_x) \, dA = 0
\]

Remark. In this early version, \( f' \) is assumed to be continuous because we need to satisfy the condition for Green's Theorem.
Lemma. Let $f$ be analytic throughout a closed region $R$ consisting of the points interior to a positively oriented simple closed contour $C$, together with the points on $C$ itself. For any positive number $\epsilon > 0$, the region $R$ can be covered with a finite number of squares and partial squares, indexed by $j = 1, 2, \ldots, n$ such that in each one there is a fixed point $z_j$ for which the inequality

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon$$

is satisfied by all points other than $z_j$ in that square or partial square.

Proof. We first construct equally spaced horizontal and vertical lines that separate the region $R$ into squares and partial squares. If in one of the squares or partial squares, we cannot find a $z_j$, then we subdivide it into four smaller squares of equal size.

We claim that after finitely times of subdivisions, each small square will satisfy the requirements of the lemma.

We'll prove by contradiction. Suppose $\Omega_k$, $k \in \mathbb{N}$ is a nested sequence of squares, the diameter of $\Omega_k$ is $\frac{b}{2^k}$, where $b$ is a constant. It can be shown that there exists a $z_0 \in \bigcap_{k=1}^{\infty} \Omega_k$. 

(61)
Since \( f \) is analytic at \( z_0 \), for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( 0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \).

But then we see if \( K \) is big enough, \( \overline{D}_K \) has diameter less than \( \varepsilon \), then for any \( z \in \overline{D}_K \), \( \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \).

Contradict to the assumption.

**Theorem (Cauchy–Goursat)**

If a function \( f \) is analytic at all points interior to and on a simple closed contour \( C \), then

\[
\oint_C f(z) \, dz = 0
\]

**Proof**

By the previous Lemma, given any \( \varepsilon > 0 \), we can cover the region \( R \) enclosed by \( C \) by squares or partial squares \( D_j \) such that \( \exists \, z_j \in D_j \), \( \forall \, z \in \overline{D}_j \setminus \{z_j\} \)

\[
\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon
\]

Define

\[\delta_j(z) = \begin{cases} \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & \text{if } z \in \overline{D}_j \setminus \{z_j\} \\ 0 & \text{if } z = z_j \end{cases}\]

Then \( \delta_j \) is a continuous function on \( \overline{D}_j \), such that \( \delta_j(z) e^{i\theta} \)

We can rewrite the above as

\[
f(z) = f(z_j) + (\delta_j(z) + f'(z_j))(z - z_j)
\]

\[
= f(z_j) - z_j f'(z_j) + f'(z)z + (z - z_j) \delta_j(z)
\]

(62)
We thus get: \( f(z) \) integrates along \( C_j \), the boundary of \( \Omega_j \) (counterclockwise)

\[
\oint_{C_j} f(z)dz = \oint_{C_j} f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j) \delta_j(z) \, dz
\]

\[
= \left( f(z_j) - z_j f'(z_j) \right) \oint_{C_j} dz + f'(z_j) \oint_{C_j} z \, dz
\]

\[
+ \oint_{C_j} (z - z_j) \delta_j(z) \, dz
\]

\[
= \oint_{C_j} (z - z_j) \delta_j(z) \, dz
\]

Another key observation is that

\[
\oint_{C_j} f(z)dz = \sum_{j} \oint_{C_j} f(z)dz = \sum_{j} \oint_{C_j} (z - z_j) \delta_j(z) \, dz
\]

If \( C_j \) is a square, let \( s_j \) be the length of an edge of \( \Omega_j \), \( A_j \) is the area of \( \Omega_j \), then

\[
\left| \oint_{C_j} (z - z_j) \delta_j(z) \, dz \right| \leq \int_{C_j} |z - z_j| |\delta_j(z)| \, dz
\]

\[
\leq (4s_j \varepsilon) \cdot (4s_j)
\]

\[
= 16s_j^2 \varepsilon A_j
\]

If \( C_j \) is the boundary of a partial square,

\[
\left| \oint_{C_j} (z - z_j) \delta_j(z) \, dz \right| \leq (4s_j \varepsilon) \cdot (4s_j + L_j)
\]

\[
< 4 \sqrt{s_j^2 \varepsilon} A_j + 4s_j \varepsilon L_j \varepsilon
\]

where \( L_j \) is the part of \( C_j \) which is also part of \( C \).
$S$ is the length of a side of some square that encloses the entire contour $C$ as well as all of the squares covering $R$. Now let

$L$ be the arc length of $C$

$$\int_{C} f(z)dz \leq \sum_{C} \int_{C} f(z)dz$$

$$\leq 4\sqrt{2}S^2 + \sqrt{2}SL \leq$$

$$= (4\sqrt{2}S^2 + \sqrt{2}SL) \epsilon$$

Since $S$ and $L$ are constants, we see as $\epsilon \to 0$, we get

$$\int_{C} f(z)dz = 0$$

Definition. A simply connected domain $D$ is a domain such that every simple closed contour within it encloses only points of $D$.

Example. $C$ is a simply connected domain.

The interior of a simple closed curve is a simply-connected domain.

The annulus region $1 < |z| < 2$ is NOT a simply-connected domain.