UPPER BOUND

Lemma. If $w(t)$ is a piecewise continuous complex valued function $w(t): [a,b] \longrightarrow \mathbb{C}$, then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$$

Proof. If $\int_a^b w(t)\,dt = 0$, trivial.

If $\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \neq 0$ we see that

$r_0 = \left|\int_a^b w(t)\,dt\right|$. So the goal is to show

$r_0 \le \int_a^b |w(t)|\,dt$.

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \Rightarrow r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$$

$$r_0 \in \mathbb{R} \Rightarrow r_0 = \int_a^b \mathrm{Re}[e^{-i\theta_0} w(t)]\,dt$$

$$\le \int_a^b |e^{-i\theta_0} w(t)|\,dt$$

$$= \int_a^b |w(t)|\,dt$$

Theorem. C is a contour. The arclength of C is $L \in \mathbb{R}^{\ge 0}$. $f(z)$ is a piecewise continuous function on C. If $M \ge 0$ such that $|f(z)| \le M$ for all points on C at which $f(z)$ is defined,

then: $\left|\int_C f(z)\,dz\right| \le M \cdot L$

Proof. For a piece of arc $C_i: z(t)$, $a \le t \le b$ in C.

By the Lemma, $\left|\int_{C_i} f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right| \le \int_a^b |f(z(t))| \cdot |z'(t)|\,dt$

Since $|f(z)| \le M$, we get

$$\int_a^b |f(z(t))| \cdot |z'(t)|\,dt \le \int_a^b M|z'(t)|\,dt = M\int_a^b |z'(t)|\,dt$$

$$= M \cdot L_i$$

A contour C is a union of several pieces of arcs $C_1, C_2, \ldots, C_n$

So $$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz \le \sum_{i=1}^{n} ML_i = M\sum_{i=1}^{n} L_i = ML.$$

Example. C_R is the semicircle $z = Re^{i\theta}$, $0 \le \theta \le \pi$

We will show $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\,dz = 0$.

Note for R big enough, by triangle inequalities

$$|z+1| \le |z| + 1 = R+1$$
$$|z^2+4| \ge |z|^2 - 4 = R^2 - 4$$
$$|z^2+9| \ge |z|^2 - 9 = R^2 - 9$$

So on C_R, $\displaystyle\left|\frac{z+1}{(z^2+4)(z^2+9)}\right| \le \frac{R+1}{(R^2-4)(R^2-9)}$

By the Theorem.

$$0 \le \left|\int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\,dz\right| \le \frac{R+1}{(R^2-4)(R^2-9)} \cdot \pi R = \pi \cdot \frac{R^2+R}{(R^2-4)(R^2-9)} = \pi \cdot \frac{1+\frac{1}{R}}{(1-\frac{4}{R^2})(R^2-9)}$$

$$0 \le \lim_{R\to\infty} \left|\int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\,dz\right| \le \lim_{R\to\infty} \pi \frac{1+\frac{1}{R}}{(1-\frac{4}{R^2})(R^2-9)} = 0$$

We conclude $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)}\,dz = 0$

ANTIDERIVATIVE

Definition. $f(z)$ is a complex function. Define the antiderivative of $f(z)$ to be $F(z)$ if $F'(z) = f(z)$.

Example. If $f(z) = z^2$, then $F(z) = \frac{1}{3}z^3$ is an antiderivative of $f(z)$

Proposition If $f(z)$ is defined on a domain D, then the antiderivative of f on D is unique up to adding a constant.

Proof. If $F(z)$, $G(z)$ are both antiderivatives of $f(z)$ then $F'(z) = f(z)$ and $G'(z) = f(z)$

So $(F(z) - G(z))' = f(z) - f(z) = 0$ on D.

We know this implies $F(z) - G(z) \equiv C \in \mathbb{C}$ on D.

It's an interesting question to ask what functions have antiderivative. It turns out we can get equivalent contiditions by studying contour integrals.

Theorem. $f(z)$ is a continuous function on a domain D. The following are equivalent:

(a). $f(z)$ has antiderivative $F(z)$ on D.

(b). If C_1 and C_2 are two contours in D with same starting point and same terminal point, then
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

(c). If C is a closed contour in D, then
$$\int_C f(z)\,dz = 0.$$

Proof. $(a) \Rightarrow (b)$

If $F'(z) = f(z)$, C is a contour in D from $z_1 \in D$ to $z_2 \in D$. We can parameterize C by some $z(t)$, $a \le t \le b$ (so $z(a) = z_1$, $z(b) = z_2$)

Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt} F(z(t))\,dt = F(z(b)) - F(z(a))$$
$$= F(z_2) - F(z_1)$$

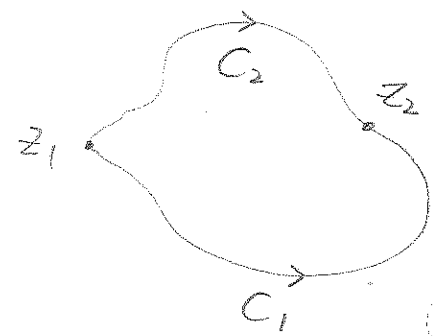
So we see $\int_C f(z)\,dz$ only depends on $F(z_2)$ and $F(z_1)$ i.e. the endpoints. If C_1 and C_2 both start at z_1 and terminates at z_2.

$$\int_{C_1} f(z)\,dz = F(z_2) - F(z_1) = \int_{C_2} f(z)\,dz$$

$(b) \Rightarrow (c)$

If C is a closed curve in D described by $z(t)$, $a \le t \le b$. Let $z_1 = z(a) = z(b)$ and $z_2 = z(c)$ for some $c \in (a, b)$. Let C_1 be the part of C from z_1 to z_2. i.e. C_1 is $z(t)$, $a \le t \le c$ and C_2 be the reversed path of z_2 to z_1 along C, i.e. C_2 is $z(-t)$, $-c \le t \le -b$.

Then $C = C_1 - C_2$ and C_1, C_2 both start at z_1 and terminates at z_2. So by assumption



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$(c) \Rightarrow (b)$. If $\int_C f(z)\,dz = 0$ for any closed path C in D

then for any C_1, C_2 both start at z_1 and terminate at z_2. $C_1 - C_2$ is a closed path in C, so

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz = 0.$$

we get $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

$(b) \Rightarrow (a)$:

Fix a point $z_0 \in D$. define

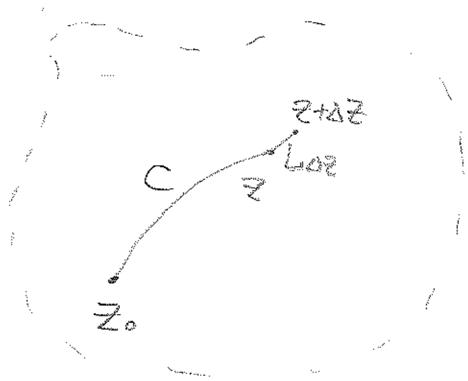
$$F(z) = \int_C f(z)\,dz \quad \text{wher } C \text{ is any contour from } z_0 \text{ to } z.$$

It's well-defined by the assumtion that all contour integrals only depend on the endpoints.

We need to verify $F'(z) = f(z)$

By definition, $F'(z) = \lim_{\Delta z \to 0} \dfrac{F(z+\Delta z) - F(z)}{\Delta z}$

We take $|\Delta z|$ to be small so that the line segment $L_{\Delta z}$ from z to $z+\Delta z$ is in D.



Then $F(z+\Delta z) - F(z)$

$$= \int_{C+L_{\Delta z}} f(z)\,dz - \int_C f(z)\,dz$$

$$= \int_{L_{\Delta z}} f(z)\,dz$$

$$= \int_0^1 f(z + t\Delta z) \cdot \Delta z \, dt$$

(Note $L_{\Delta z}$ can be expressed as $z(t) = z + t\Delta z$, $0 \le t \le 1$)

$$= \Delta z \int_0^1 f(z + t\Delta z)\,dt$$

This implies $\frac{F(z+\Delta z)-F(z)}{\Delta z} = \int_0^1 f(z+t\Delta z)\,dt$

We have assumed $f(z)$ is a continuous function, so for any $\varepsilon>0$, $\exists\, \delta>0$ such that $|\Delta z|<\delta \Rightarrow |f(z+\Delta z)-f(z)|<\varepsilon$

Then $\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right|$

$$= \left|\int_0^1 f(z+t\Delta z)\,dt - \int_0^1 f(z)\,dt\right|$$

$$\leq \int_0^1 |f(z+t\Delta z)-f(z)|\,dt$$

$$< \int_0^1 \varepsilon\,dt$$

$$= \varepsilon \quad \text{for any } 0<|\Delta z|<\delta.$$

So we conclude $\lim_{\Delta z\to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$

i.e. $F'(z)=f(z)$.

Corollary. If $F(z)$ is the antiderivative of $f(z)$, then for any contour C from z_1 to z_2 in a domain D,

$$\int_C f(z)\,dz = F(z_2)-F(z_1)$$

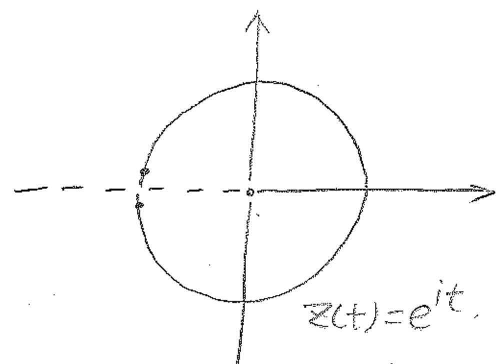
We've proved it during the proof of the theorem (Part (a)⇒(b))

Example. If C is the circle $z(t)=e^{it}$, $0\le t\le 2\pi$,
then $\int_C \frac{1}{z^2}\,dz = 0$ since $\frac{1}{z^2}$ has antiderivative $-\frac{1}{z}$ on the domain $\mathbb{C}\setminus\{0\}$.

Note: But $\int_C \frac{1}{z}dz \neq 0$ since $\frac{1}{z}$ doesn't have antiderivative on $\mathbb{C}\setminus\{0\}$: Recall that $(\log z)' = \frac{1}{z}$ for any branch of $\log z$, but we need to make a branch cut to get a branch, so $\log z$ is not a single-valued analytic function on $\mathbb{C}\setminus\{0\}$. it cannot be used as the antiderivative of $\frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$.

However, we can use the limit trick to apply the antiderivative method:

We take the branch of $\log z$: $-\pi < \theta < \pi$, then $\log z$ is the antiderivative of $\frac{1}{z}$ on this domain $\mathbb{C}\setminus\mathbb{R}^{\leq 0}$.



$$\int_C \frac{1}{z}dz = \lim_{\varepsilon\to 0}\int_{z(-\pi+\varepsilon)}^{z(\pi-\varepsilon)} \frac{1}{z}\,dz$$

$$= \lim_{\varepsilon\to 0} \log(z(\pi-\varepsilon)) - \log(z(-\pi+\varepsilon))$$

$$= \lim_{\varepsilon\to 0} i(\pi-\varepsilon) - i(-\pi+\varepsilon)$$

$$= 2\pi i$$