Lemma. If \( w(t) \) is a piecewise continuous complex valued function \( w(t): [a, b] \rightarrow \mathbb{C} \), then
\[
\left| \int_a^b w(t) \, dt \right| \leq \int_a^b |w(t)| \, dt
\]

Proof. If \( \int_a^b w(t) \, dt = 0 \), trivial.

If \( \int_a^b w(t) \, dt = R_0 e^{i\theta_0} \neq 0 \), we see that
\[
R_0 = \left| \int_a^b w(t) \, dt \right|.
\]
So the goal is to show
\[
R_0 \leq \int_a^b |w(t)| \, dt.
\]

\[
\int_a^b w(t) \, dt = R_0 e^{i\theta_0} \Rightarrow R_0 = \int_a^b e^{-i\theta_0} w(t) \, dt
\]
\[
r_0 \in \mathbb{R} \Rightarrow R_0 = \int_a^b \text{Re}\{e^{i\theta_0} w(t)\} \, dt = \int_a^b |e^{-i\theta_0} w(t)| \, dt = \int_a^b |w(t)| \, dt
\]

Theorem. \( C \) is a contour. The arclength of \( C \) is \( L \in \mathbb{R}^{\geq 0} \), \( f(z) \) is a piecewise continuous function on \( C \). If \( M > 0 \) such that
\[
|f(z)| \leq M \text{ for all points on } C \text{ at which } f(z) \text{ is defined},
\]
then:
\[
\left| \int_C f(z) \, dz \right| \leq ML
\]

Proof. For a piece of arc \( C_t: z(t) \), \( a \leq t \leq b \), in \( C \).

By the Lemma, \[
\left| \int_{C_t} f(z) \, dz \right| \leq \int_a^b |f(z(t))| |z'(t)| \, dt \leq \int_a^b |f(z(t))| |z(t)| \, dt
\]

\[\boxed{}\]
Since $|f(z)| \leq M$, we get

$$\int_a^b f(z+1) \cdot |z+1| \, dz \leq \int_a^b M |z+1| \, dz = M \int_a^b |z+1| \, dz = M L,$$

A contour $C$ is a union of several pieces of arcs $C_1, C_2, \ldots, C_n$

So

$$\int_C f(z) \, dz = \sum_{i=1}^n \int_{C_i} f(z) \, dz \leq \sum_{i=1}^n M L_i = M \sum_{i=1}^n L_i = ML.$$

Example.

$C_R$ is the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

We will show $\lim_{R \to \infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} \, dz = 0$.

Note for $R$ big enough, by triangle inequalities,

$$|z+1| \leq |z| + 1 = R + 1$$

$$|z^2+4| \geq |z^2-4| = R^2 - 4$$

$$|z^2+9| \geq |z^2-9| = R^2 - 9$$

So on $C_R$, $|\frac{z+1}{(z^2+4)(z^2+9)}| \leq \frac{R+1}{(R^2-4)(R^2-9)}$.

By the theorem:

$$0 \leq \left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} \, dz \right| \leq \frac{R+1}{(R^2-4)(R^2-9)} \cdot \pi R = \pi \cdot \frac{R^2+R}{(R^2-4)(R^2-9)} = \pi \cdot \frac{1 + \frac{1}{R}}{(1 - \frac{1}{R})(1 - \frac{1}{R})}.$$ \[lim_{R \to \infty} \pi \cdot \frac{1 + \frac{1}{R}}{(1 - \frac{1}{R})(1 - \frac{1}{R})} = 0\]

We conclude

$$\lim_{R \to \infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} \, dz = 0.$$
Definition. \( f(z) \) is a complex function. Define the antiderivative of \( f(z) \) to be \( F(z) \) if \( F(z) = f(z) \).

Example. If \( f(z) = z^2 \), then \( F(z) = \frac{1}{3} z^3 \) is an antiderivative of \( f(z) \).

Proposition. If \( f(z) \) is defined on a domain \( D \), then the antiderivative of \( f \) on \( D \) is unique up to adding a constant.

Proof. If \( F(z) \), \( G(z) \) are both antiderivatives of \( f(z) \), then \( F(z) = f(z) \) and \( G(z) = f(z) \).

So \( (F(z) - G(z))' = f(z) - f(z) = 0 \) on \( D \).

We know this implies \( F(z) - G(z) = C \in \mathbb{C} \) on \( D \).

It's an interesting question to ask what functions have antiderivative. It turns out we can get equivalent conditions by studying contour integrals.

Theorem. \( f(z) \) is a continuous function on a domain \( D \). The following are equivalent:

(a). \( f(z) \) has antiderivative \( F(z) \) on \( D \).

(b). If \( C_1 \) and \( C_2 \) are two contours in \( D \) with same starting point and same terminal point, then 
\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz. \]

(c). If \( C \) is a closed contour in \( D \), then 
\[ \int_C f(z) \, dz = 0. \]
(a) \Rightarrow (b)

If \( F(z) = -f(z) \), \( C \) is a contour in \( D \) from \( z, \in D \) to \( \zeta, \in D \). We can parameterize \( C \) by some \( z(t) \), \( a \leq t \leq b \) (so \( z(a) = z_1, \ z(b) = z_2 \)).

Then
\[
\int_C f(z) \, dz = \int_a^b f(z(t)) \cdot z'(t) \, dt = \int_a^b \frac{d}{dt} F(z(t)) \, dt = F(z(b)) - F(z(a))
\]
\[
= F(z_2) - F(z_1)
\]

So we see \( \int_C f(z) \, dz \) only depends on \( F(z_1) \) and \( F(z_2) \), i.e. the endpoints. If \( C_1 \) and \( C_2 \) both start at \( z_1 \) and terminates at \( z_2 \),

\[
\int_{C_1} f(z) \, dz = F(z_2) - F(z_1) = \int_{C_2} f(z) \, dz.
\]

(b) \Rightarrow (c)

If \( C \) is a closed curve in \( D \), described by \( z(t), a \leq t \leq b \). Let \( z_1 = z(a) = z(b) \) and \( z_2 = z(c) \) for some \( c \in (a, b) \). Let \( C_1 \) be the part of \( C \) from \( z_1 \) to \( z_2 \), i.e. \( C_1 \) is \( z(t), a \leq t \leq c \) and \( C_2 \) be the reversed path of \( z_2 \) to \( z_1 \) along \( C \), i.e. \( C_2 \) is \( z(t), -c \leq t \leq -a \).

Then \( C = C_1 - C_2 \). and \( C_1 \), \( C_2 \) both start at \( z_1 \) and terminates at \( z_2 \). So by assumption,

\[
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz
\]

\[
\int_C f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz = 0.
\]
(c) \Rightarrow (b). If \( \int_C f(z) \, dz = 0 \) for any closed path \( C \) in \( D \) then for any \( C_1, C_2 \) both start at \( z \), and terminate at \( z \), \( C_1 - C_2 \) is a closed path in \( C \). So

\[
\int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz = \int_{C_1 - C_2} f(z) \, dz = 0.
\]

We get \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \).

(b) \Rightarrow (a):

Fix a point \( z_0 \in D \), define

\[ F(z) = \int_C f(z) \, dz \text{ when } C \text{ is any contour from } z_0 \text{ to } z. \]

It's well-defined by the assumption that all contour integrals only depend on the endpoints.

We need to verify \( F(z) = f(z) \).

By definition, \( F(z) = \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} \).

We take \( |\Delta z| \) to be small so that the line segment \( L_{\Delta z} \) from \( z \) to \( z + \Delta z \) is in \( D \).

Then \( F(z + \Delta z) - F(z) \)

\[
= \int_{C + L_{\Delta z}} f(z) \, dz - \int_C f(z) \, dz
\]

\[
= \int_{L_{\Delta z}} f(z) \, dz.
\]

\[
= \int_0^1 f(z + t\Delta z) \cdot \Delta z \, dt
\]

(Note \( L_{\Delta z} \) can be expressed as \( z(t) = z + t\Delta z, 0 \leq t \leq 1 \))
This implies \( \frac{F(z+\Delta z) - F(z)}{\Delta z} = \int_0^1 f(z+\tau \Delta z) \, d\tau \).

We have assumed \( f(z) \) is a continuous function, so for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( |\Delta z| < \delta \Rightarrow |f(z+\Delta z) - f(z)| < \varepsilon \).

Then \[
\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right|
\leq |\int_0^1 f(z+\tau \Delta z) - f(z) \, d\tau|
\leq \int_0^1 |f(z+\tau \Delta z) - f(z)| \, d\tau
\leq \varepsilon \int_0^1 \tau \, d\tau
= \varepsilon \quad \text{for any} \quad 0 < |\Delta z| < \delta.
\]

So we conclude \( \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z) \).

\[ \text{I.e.,} \quad F'(z) = f(z). \]

**Corollary.** If \( F(z) \) is the antiderivative of \( f(z) \), then for any contour \( C \) from \( z_1 \) to \( z_2 \) in a domain \( D \),
\[
\int_C f(z) \, dz = F(z_2) - F(z_1).
\]

We've proved it during the proof of the theorem. (Part (a) \( \Rightarrow \) (b))

**Example.** If \( C \) is the circle \( z(t) = e^{\pi t}, \ a < \pi r < b \),

then \( \int_C \frac{1}{z^2} \, dz = 0 \) since \( \frac{1}{z^2} \) has antiderivative \( -\frac{1}{z} \) on the domain \( C \setminus \{0\} \).
Note: But $\int_C \frac{1}{z} \, dz \neq 0$ since $\frac{1}{z}$ does not have an antiderivative on $C \setminus \{0\}$: Recall that $(\log z)' = \frac{1}{z}$ for any branch of $\log z$, but we need to make a branch cut to get a branch, so $\log z$ is not a single-valued analytic function on $C \setminus \{0\}$, it cannot be used as the antiderivative of $\frac{1}{z}$ on $C \setminus \{0\}$.

However, we can use the limit trick to apply the antiderivative method:

We take the branch of $\log z$: $-\pi < \theta < \pi$, then $\log z$ is the antiderivative of $\frac{1}{z}$ on this domain $\mathbb{C} \setminus \{0\}$.

$$\int_C \frac{1}{z} \, dz = \lim_{\varepsilon \to 0} \int_{\gamma(-\pi + \varepsilon)}^{\gamma(\pi - \varepsilon)} \frac{1}{z} \, dz$$

$$= \lim_{\varepsilon \to 0} \log(z(\pi - \varepsilon)) - \log(z(-\pi + \varepsilon))$$

$$= \lim_{\varepsilon \to 0} i(\pi - \varepsilon) - i(-\pi + \varepsilon)$$

$$= 2\pi i$$