COMPLEX FUNCTIONS

In the subject of Complex Analysis, we are mostly interested in functions $f: \mathbb{C} \longrightarrow \mathbb{C}$, or in some cases the domain is not all $\mathbb{C}$, but a subset of it.

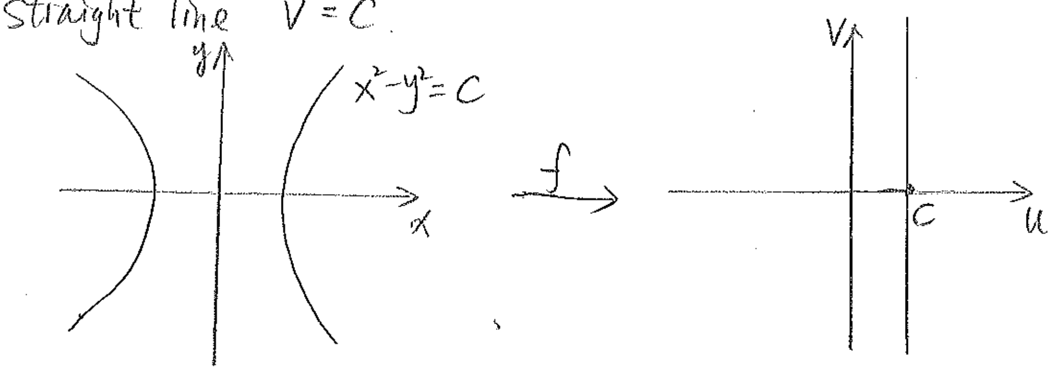
We will take $w = z^2$ as a first example of functions on complex numbers.

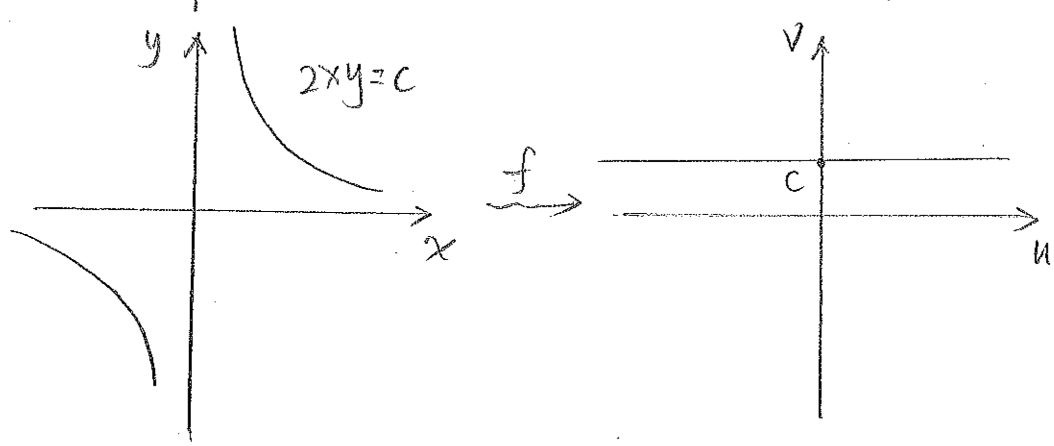
Define $f: \mathbb{C} \longrightarrow \mathbb{C}$, the quadratic function.
$z \longmapsto z^2$.

If $w = u + vi = f(z) = f(x+yi) = (x+yi)^2 = (x^2-y^2) + 2xyi$

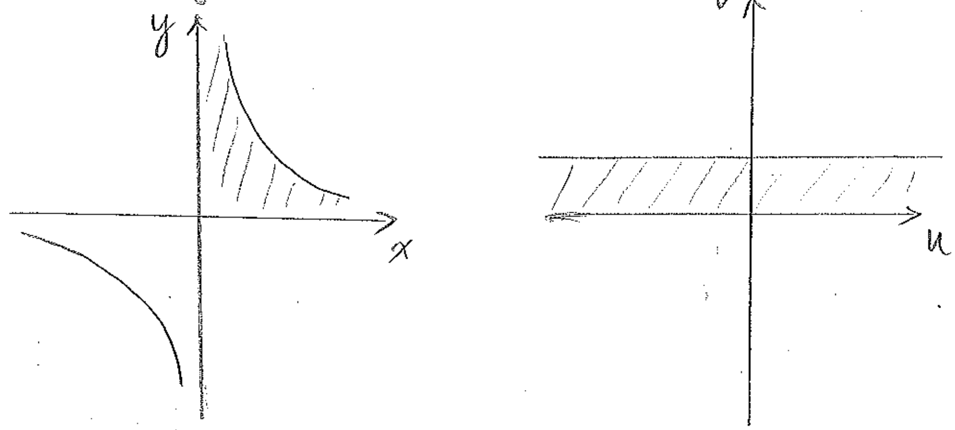
we see $u = x^2 - y^2$ and $v = 2xy$

This implies the function $w = z^2$ sends a curve $x^2-y^2 = c$ on the complex plane to the straight line $u = c$, and sends the curve $2xy = c$ on the complex plane to the straight line $v = c$.





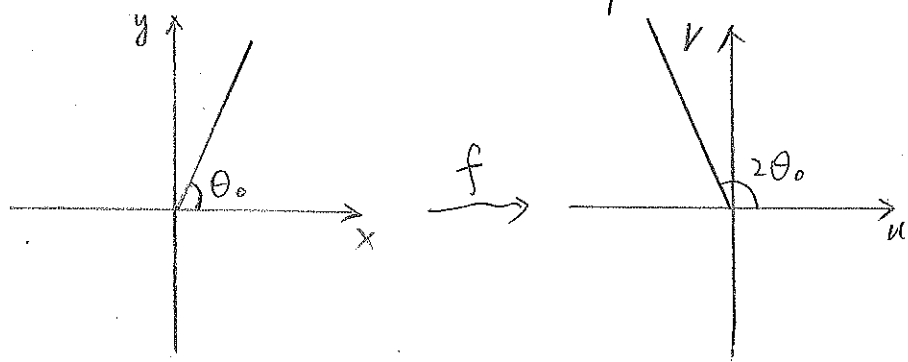
More generally, $w = z^2$ transforms a region in xy-plane to a region in uv-plane:



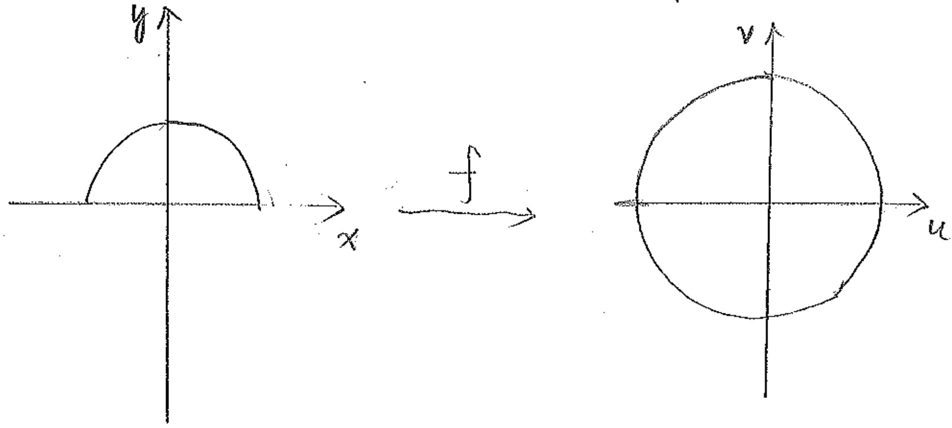
Now if we write the function in the exponential form, we see

$$w = z^2 = (re^{i\theta})^2 = r^2 e^{2\theta i}$$

This indicates each ray $\theta = \theta_0$ on xy-plane is sent to $\theta = 2\theta_0$ on the uv-plane:



and sends the Hemicircle $r = R$, $0 \le \theta \le \pi$ to the circle $r = R^2$.



LIMIT AND CONTINUITY.

We know in Analysis, the foundation of the whole subject is the concept of limit. This definition can be extened to complex functions.

Definition. f is a complex valued function defined at all points z in some deleted neighbourhood of a point $z_0 \in \mathbb{C}$. Define $\lim_{z \to z_0} f(z) = w_0$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon.$$

Intuitively, it means as z approaches z_0, $f(z)$ approaches w_0

Theorem. If the limit $\lim_{z \to z_0} f(z)$ exists, it is unique.

Proof. Suppose $w_0 \neq w_1 \in \mathbb{C}$ are both limits $\lim_{z \to z_0} f(z)$,

i.e. $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$

Let $\varepsilon = \frac{1}{2}|w_0 - w_1| > 0$. there exists $\delta_0 > 0$ and $\delta_1 > 0$

such that $\begin{cases} 0 < |z - z_0| < \delta_0 \Rightarrow |f(z) - w_0| < \varepsilon \\ 0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - w_1| < \varepsilon \end{cases}$

so when $0 < |z - z_0| < \min\{\delta_0, \delta_1\}$.

$$|w_0 - w_1| \leq |w_0 - f(z)| + |f(z) - w_1| < \varepsilon + \varepsilon = 2\varepsilon = |w_0 - w_1|$$

contradiction.

Example. Let $f(z) = \frac{iz}{2}$. We can show that $\lim_{z \to 1} f(z) = \frac{i}{2}$:

For any $\varepsilon > 0$, let $\delta = 2\varepsilon > 0$

For any $0 < |z - 1| < \delta = 2\varepsilon$, $|f(z) - \frac{i}{2}| = |\frac{iz}{2} - \frac{i}{2}| = \frac{1}{2}|z - 1| < \varepsilon$

Example $f(z) = \frac{z}{\bar{z}}$. We can show the limit $\lim_{z\to 0} f(z)$ doesn't exist:

If z approaches 0 from positive real axis,

$$f(z) = \frac{x+0i}{x-0i} = \frac{x}{x} = 1$$

If z approaches 0 from positive imaginary axis,

$$f(z) = \frac{0+yi}{0-yi} = \frac{yi}{-yi} = -1$$

So we see as z approach 0 from different paths, the $f(z)$ converges to different values, it's impossible for $\lim_{z\to 0} f(z)$ to exist.

Theorem. If $f(z) = f(x+iy) = u(x,y) + v(x,y)i$, then

$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ if and only if

$$\lim_{z\to x_0+y_0 i} f(z) = u_0 + v_0 i.$$

Proof. "$\Rightarrow$": For any $\varepsilon>0$, $\exists \delta_1>0$, $\delta_2>0$ such that

$$\begin{cases} |(x,y)-(x_0,y_0)| < \delta_1 \Rightarrow |u(x,y)-u_0| < \frac{\varepsilon}{2} \\ |(x,y)-(x_0,y_0)| < \delta_2 \Rightarrow |v(x,y)-v_0| < \frac{\varepsilon}{2} \end{cases}$$

Then for any $|(x+yi)-(x_0+y_0 i)| = \sqrt{(x-x_0)^2+(y-y_0)^2} < \min\{\delta_1, \delta_2\}$

$$\begin{cases} |x-x_0| < \sqrt{(x-x_0)^2+(y-y_0)^2} < \min\{\delta_1,\delta_2\} \\ |y-y_0| < \sqrt{(x-x_0)^2+(y-y_0)^2} < \min\{\delta_1,\delta_2\} \end{cases}$$

So $|u(x,y)-u_0| < \frac{\varepsilon}{2}$, $|v(x,y)-v_0| < \frac{\varepsilon}{2}$

$$|f(x+yi)-(u_0+v_0 i)| = |u(x,y)+v(x,y)i-u_0-v_0 i|$$
$$\leq |u(x,y)-u_0| + |v(x,y)-v_0| < \varepsilon$$

"$\Leftarrow$": If $\lim_{z\to z_0} f(z) = u_0 + v_0 i$.

Then for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < |(x+yi) - (x_0 + y_0 i)| < \delta \Rightarrow |u(x,y) + v(x,y)i - (u_0 + v_0 i)| < \varepsilon$$

Note: $|(x,y) - (x_0, y_0)| = |(x+yi) - (x_0 + y_0 i)|$.

so $\forall\ 0 < |(x,y) - (x_0, y_0)| < \delta$.

$$|u(x,y) - u_0| \le |(u(x,y) - u_0) + (v(x,y) - v_0)i| < \varepsilon$$

$$|v(x,y) - v_0| \le |(u(x,y) - u_0) + (v(x,y) - v_0)i| < \varepsilon$$

Theorem. If $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} g(z) = w_2$, then

(i) $\lim_{z\to z_0} [f(z) \pm g(z)] = w_1 \pm w_2$.

(ii). $\lim_{z\to z_0} [f(z)\, g(z)] = w_1 w_2$

(iii) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{w_1}{w_2}$ if $w_2 \neq 0$.

Corollary. If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial in $\mathbb{C}[z]$, then $\lim_{z\to z_0} P(z) = P(z_0)$

Sometimes we are interested in the behavior of $f(z)$ as $|z| \to \infty$.

First, we can add a "point of infinity" to the complex plane in a very natural way, by constructing the famous Riemann Sphere.