EXPONENTIAL FORM OF COMPLEX NUMBERS

Recall that we can identify the complex number \( z = x + yi \) with the point \((x, y)\) on the plane. Given a point \((x, y)\) in the Cartesian plane, we can describe it in the polar coordinates by \((r, \theta)\), where \(r\) is the distance between \((x, y)\) and the origin, \(r = \sqrt{x^2 + y^2}\), and \(\theta\) is the angle whose terminal edge passes through \((x, y)\), so \(x = r\cos \theta, y = r \sin \theta\). Note that the choice of \(\theta\) is not unique, and you can add integral multiples of \(2\pi\) to \(\theta\) to represent the same terminal edge.

This observation motivates us to represent a complex number by polar coordinates: if \( z = x + yi \), we write \(z = r\cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)\)

where \((r, \theta)\) is the polar coordinate corresponding to the Cartesian coordinate \((x, y)\).

Definition. We call \(\theta\) an argument of \(z\), and \(\text{arg} \, z\) is the set of all such \(\theta\). We define the principal value of \(\text{arg} \, z\) to be the value in \(\text{arg} \, z \cap (-\pi, \pi]\), and denote by \(\text{Arg} \, z\).

Remark. \(\text{arg} \, z = \{\text{Arg} \, z + 2\pi k \in \mathbb{R} | k \in \mathbb{Z}\}\)
Example. If $z = 1 + i$, $\text{Arg } z = \frac{\pi}{4}$.
If $z = 1 - i$, $\text{Arg } z = -\frac{\pi}{4}$

Exercise. Find $\text{Arg } z$ if $z = 1, i, -1, -i$

Definition. We write $e^{i\theta} = \cos \theta + i\sin \theta$, so $re^{i\theta} = r\cos \theta + ir\sin \theta$

Remark. 1. This notation will be natural if you consider the Taylor Expansion of $e^x$, $\cos x$, $\sin x$.

2. An interesting special case is to take $\theta = \pi$: we get $e^{i\pi} = \cos \pi + i\sin \pi \Rightarrow e^{i\pi} = 1 + 0$. So we obtain $e^{i\pi} + 1 = 0$

This is the famous Euler's formula, and is regarded as one of the most beautiful formulae in history.

With this notation, we can write a complex number $z$ as $z = re^{i\theta}$, where $r = |z|$ and $\theta \in \text{Arg } z$.

Example. $1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$
$1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$

One advantage of this notation is that it brings a convenient way to compute the product of complex numbers:

Lemma. $(re^{i\alpha})(r_2e^{i\beta}) = rr_2e^{i(\alpha + \beta)}$

Proof. Recall $e^{i\theta} = \cos \theta + i\sin \theta$. 

(6)
\[ (r_1 \, e^{i\theta_1})(r_2 \, e^{i\theta_2}) = r_1 \, (\cos \theta_1 + i \sin \theta_1) \, r_2 \, (\cos \theta_2 + i \sin \theta_2) \]
\[ = r_1 \, r_2 \, [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)] \]
\[ = r_1 \, r_2 \, [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \]
\[ = r_1 \, r_2 \, e^{i(\theta_1 + \theta_2)} \]

**Example:**
\[ (1 + i)(1 - i) = \sqrt{2} \, e^{\frac{\pi}{4}i} \cdot \sqrt{2} \, e^{\frac{-\pi}{4}i} = (\sqrt{2})^2 \cdot e^{\left(\frac{\pi}{4} - \frac{\pi}{4}\right)i} = 2 \cdot e^0 = 2(\cos 0 + i \sin 0) = 2 \]

**Corollary.**
(i) \( (r \, e^{i\theta})^{-1} = r^{-1} e^{-i\theta} \)
(ii) \( \frac{r_1 \, e^{i\theta_1}}{r_2 \, e^{i\theta_2}} = \frac{r_1}{r_2} \, e^{i(\theta_1 - \theta_2)} \) \( (r_2 \neq 0) \)
(iii) \( (r \, e^{i\theta})^n = r^n \, e^{i\theta} \) for any \( n \in \mathbb{Z} \).

**Proof.**
(i) \( (r^{-1} e^{-i\theta}) \cdot (r \, e^{i\theta}) = (r^{-1} \cdot r) \, e^{i(-\theta + \theta)} = 1 \)

So \( r^{-1} e^{i\theta} = (r \, e^{i\theta})^{-1} \)

(ii) \( \frac{r_1 \, e^{i\theta_1}}{r_2 \, e^{i\theta_2}} = (r_1 \, e^{i\theta_1}) \cdot (r_2 \, e^{-i\theta_2})^{-1} = r_1 \, e^{i\theta_1} \cdot r_2^{-1} \, e^{-i\theta_2} \)

\[ = r_1 \, r_2^{-1} \, e^{i(\theta_1 - \theta_2)} \]

(iii) When \( n = 0 \), \( (r \, e^{i\theta})^0 = 1 \) by default, and \( r^0 e^{i\theta} = 1 \).

When \( n > 0 \), we can prove by induction:
1. \( n = 1 : (r \, e^{i\theta})^1 = r \, e^{i\theta} \)
2. Assume \( (r \, e^{i\theta})^n = r^n \, e^{i\theta} \), then
\[ (r \, e^{i\theta})^{n+1} = (r \, e^{i\theta})^n \cdot (r \, e^{i\theta}) = r^n \, e^{i\theta} \cdot r \, e^{i\theta} \]
\[ = r^{n+1} \, e^{i(n\theta + \theta)} \]
\[ = r^{n+1} \, e^{i(n+1)\theta} \]

When \( n < 0 \), \( (r \, e^{i\theta})^n = [r^{-1} e^{-i\theta}]^{-n} \), then apply the previous case to \(-n > 0\).
Corollary (Morrie's Formula) \((\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\)

Example \((e^{i\theta})^2 = e^{i2\theta}\)

\((\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta\)

\((\cos^2 \theta - \sin^2 \theta) + i \cdot 2 \sin \theta \cos \theta = \cos 2\theta + i \sin 2\theta\), we therefore have

\[
\begin{cases}
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \\
\sin 2\theta = 2 \sin \theta \cos \theta
\end{cases}
\]

so some of the trigonometric identities we've seen before can be reproved by complex numbers.

The formula \(r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}\) indicates the following:

1. \(|z_1 z_2| = |z_1| \cdot |z_2|\)
2. \(\theta_1, \in \text{Arg} z_1, \theta_2, \in \text{Arg} z_2 \Rightarrow \theta_1 + \theta_2 \in \text{Arg}(z_1 z_2)\)

Example. \(\text{Arg} z_1 = \frac{\pi}{3}, \text{Arg} z_2 = \frac{3}{4} \pi\). What is \(\text{Arg}(z_1 z_2)\)?

\[
\frac{\pi}{3} + \frac{3}{4} \pi = \frac{13}{12} \pi \in \text{Arg}(z_1 z_2), \quad \frac{13}{12} \pi - 2\pi = -\frac{11}{12} \pi \in (-\pi, \pi]
\]

So \(\text{Arg}(z_1 z_2) = -\frac{11}{12} \pi\)
$n$-th Roots of a Complex Number

An interesting application of the exponential form is to use it to compute the $n$-th roots of a given complex number.

Given a complex number $C$, we would like to find all complex numbers $Z$ such that $Z^n = C$, where $n$ is a positive integer.

A first special case is $C=0$. In this case, $Z^n = 0$, this implies $|Z^n| = 0 \Rightarrow |Z|^n = 0 \Rightarrow |Z| = 0 \Rightarrow Z = 0$. So $Z = 0$ is the only solution.

A more interesting case is $C = 1$. If $Z^n = 1$, we say $Z$ is an $n$-th roots of unity.

Recall that $\text{arg}(1) = \{z \in \mathbb{R} | k \in \mathbb{Z}\}$, so if we write $Z = r e^{i \theta}$, we see $(r e^{i \theta})^n = e^{i n \theta}$ for some $k \in \mathbb{Z}$, i.e.

$$r^n e^{i n \theta} = e^{2k \pi i}$$

This implies $r^n = 1$, so $r = 1$.

$$n \theta = 2k \pi \Rightarrow \theta = \frac{2k \pi}{n}$$

and conversely, if $\theta = \frac{2k \pi}{n}$ for some $k \in \mathbb{Z}$,

$$e^{i \theta} = e^{\frac{2k \pi i}{n}} = e^{2k \pi i} = 1$$

We conclude the set of $n$-th roots of unity are

$$\{e^{\frac{2k \pi i}{n}} | 0 \leq k \leq n - 1\}$$

Geometrically, they're evenly distributed on the unit circle.

The case $n = 6$
More generally, we can follow the same idea:

If \( C = R e^{i\theta} \) for some \( R > 0, \theta \in \mathbb{R} \).

The solutions of \( z^n = C \) are

\[
\left\{ \sqrt[n]{R} e^{\frac{\theta + 2k\pi}{n}} \in \mathbb{C} \mid 0 \leq k < n-1 \right\}
\]

So there are always \( n \) solutions to \( z^n = C \), where \( C \) is a nonzero complex number.

The \( n \) solutions are called the \( n \)-th roots of \( C \), they're of same norm, and evenly distributed on the circle \( \|z\| = R^{\frac{1}{n}} \).

Example. Solve \( z^4 = i \)

\[
i = e^{\frac{\pi}{2} + 2k\pi i}, \quad k \in \mathbb{Z}, \quad \text{so} \quad z = e^{\frac{\pi}{4} + \frac{2k\pi}{4} i}
\]

the roots are: \( e^{\frac{\pi}{4} i}, e^{\frac{5\pi}{4} i}, e^{\frac{9\pi}{4} i}, e^{\frac{13\pi}{4} i} \).