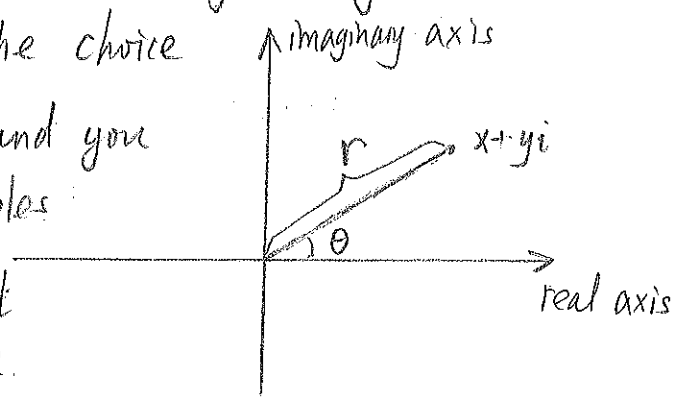
EXPONENTIAL FORM OF COMPLEX NUMBERS

Recall that we can identify the complex number $z = x+yi$ with the point (x, y) on the plane. Given a point (x, y) in the Cartisian plane, we can describe it in the polar coordinates by (r, θ), where r is the distance between (x, y) and the origin, $r = \sqrt{x^2+y^2}$, and θ is the angle whose terminal edge passes through (x, y), so $x = r\cos\theta$, $y = r\sin\theta$. Note that the choice of θ is not unique, and you can add integral multiples of 2π to θ to represent the same terminal edge.



This observation motivates us to represent a complex number by polar coordinates: If $z = x+yi$, we write

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

where (r, θ) is the polar coordinate corresponding to the Cartisian coordinate (x, y)

Definition We call θ an argument of z, and $\arg z$ is the set of all such θ. We define the principal value of $\arg z$ to be the value in $\arg z \cap (-\pi, \pi]$, and denote by $\operatorname{Arg} z$.

Remark. $\arg z = \{\operatorname{Arg} z + 2\pi k \in \mathbb{R} \mid k \in \mathbb{Z}\}$

Example. If $z = 1+i$, $\operatorname{Arg} z = \frac{\pi}{4}$.

If $z = 1-i$, $\operatorname{Arg} z = -\frac{\pi}{4}$

Exercise. Find $\operatorname{Arg} z$ if $z = 1, i, -1, -i$.

Definition. We write $e^{i\theta} = \cos\theta + i\sin\theta$, so $re^{i\theta} = r\cos\theta + ir\sin\theta$

Remark. ① This notation will be natural if you consider the Taylor Expansion of e^x, $\cos x$, $\sin x$.

② An interesting special case is to take $\theta = \pi$: we get $e^{i\pi} = \cos\pi + i\sin\pi \Rightarrow e^{i\pi} = 1+0$. so we obtain

$$\boxed{e^{i\pi} + 1 = 0}$$

This is the famous Euler's formula, and is regarded as one of the most beautiful formulae in history.

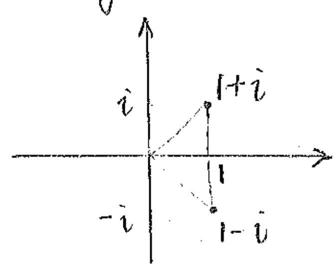
With this notation, we can write a complex number z as

$$z = re^{i\theta}.$$

where $r = |z|$ and $\theta \in \arg z$.

Example. $1+i = \sqrt{2}\, e^{\frac{\pi}{4}i}$

$1-i = \sqrt{2}\, e^{-\frac{\pi}{4}i}$



One advantage of this notation is that it brings a convenient way to compute the product of complex numbers:

Lemma. $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2\, e^{i(\theta_1+\theta_2)}$

Proof. Recall $e^{i\theta} = \cos\theta + i\sin\theta$.

$$\text{so } (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right]$$
$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$
$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Example. $(1+i)(1-i) = \sqrt{2}e^{\frac{\pi}{4}i}\cdot\sqrt{2}e^{-\frac{\pi}{4}i} = (\sqrt{2})^2\cdot e^{(\frac{\pi}{4}-\frac{\pi}{4})i} = 2\cdot e^{0\cdot i} = 2(\cos 0 + i\sin 0)$
$= 2$

Corollary. (i) $(re^{i\theta})^{-1} = r^{-1}e^{-i\theta}$

(ii) $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2}\cdot e^{i(\theta_1-\theta_2)}$ $\quad (r_2 \neq 0)$

(iii) $(re^{i\theta})^n = r^n e^{in\theta}$ for any $n\in\mathbb{Z}$.

Proof. (i) $(r^{-1}e^{-i\theta})\cdot(re^{i\theta}) = (r^{-1}r)e^{i(-\theta+\theta)} = 1$

so $r^{-1}e^{-i\theta} = (re^{i\theta})^{-1}$

(ii) $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})^{-1} = r_1 e^{i\theta_1}\, r_2^{-1} e^{-i\theta_2}$
$= r_1 r_2^{-1} e^{i(\theta_1-\theta_2)}$

(iii) When $n=0$, $(re^{i\theta})^0 = 1$ by default, and $r^0 e^{i0} = 1$.

when $n>0$, we can prove by induction:
① $n=1$: $(re^{i\theta})^1 = r^1 e^{i\theta}$
② Assume $(re^{i\theta})^n = r^n e^{in\theta}$, then

$$(re^{i\theta})^{n+1} = (re^{i\theta})^n\cdot(re^{i\theta}) = r^n e^{in\theta}\, re^{i\theta}$$
$$= r^{n+1} e^{i(n\theta+\theta)}$$
$$= r^{n+1} e^{i(n+1)\theta}$$

when $n<0$, $(re^{i\theta})^n = [r^{-1}e^{i(-\theta)}]^{-n}$, then apply the previous case to $-n>0$.

Corollary (Moivre's Formula) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Example $(e^{i\theta})^2 = e^{i\cdot 2\theta}$

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$(\cos^2\theta - \sin^2\theta) + i\cdot 2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$, we therefore have

$\begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{cases}$ so some of the trigonometric identities we've seen before can be reproved by complex numbers.

The formula $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$ indicates the following:

(1). $|z_1 z_2| = |z_1| \cdot |z_2|$
(2). $\theta_1 \in \arg z_1$, $\theta_2 \in \arg z_2 \Rightarrow \theta_1 + \theta_2 \in \arg(z_1 z_2)$

Example. $\operatorname{Arg} z_1 = \frac{\pi}{3}$, $\operatorname{Arg} z_2 = \frac{3}{4}\pi$. what is $\operatorname{Arg}(z_1 z_2)$?

$\frac{\pi}{3} + \frac{3}{4}\pi = \frac{13}{12}\pi \in \arg(z_1 z_2)$, $\frac{13}{12}\pi - 2\pi = -\frac{11}{12}\pi \in (-\pi, \pi]$.

so $\operatorname{Arg}(z_1 z_2) = -\frac{11}{12}\pi$

n-th Roots of a Complex Number

An interesting application of the exponential form is to use it to compute the n-th roots of a given complex number.

Given a complex number c, we would like to find all complex numbers z such that $z^n = c$, where n is a positive integer.

A first special case is $c=0$: In this case, $z^n = 0$, this implies $|z^n| = 0 \Rightarrow |z|^n = 0 \Rightarrow |z| = 0 \Rightarrow z = 0$ so $z=0$ is the only solution.

A more interesting case is $c=1$: $z^n = 1$
If $z^n=1$, we say z is an n-th roots of unity.

Recall that $\arg(1) = \{2k\pi \in \mathbb{R} \mid k \in \mathbb{Z}\}$, so if we write $z = re^{i\theta}$, we see $(r \cdot e^{i\theta})^n = e^{2k\pi i}$ for some $k \in \mathbb{Z}$

i.e. $r^n e^{in\theta} = e^{2k\pi i}$

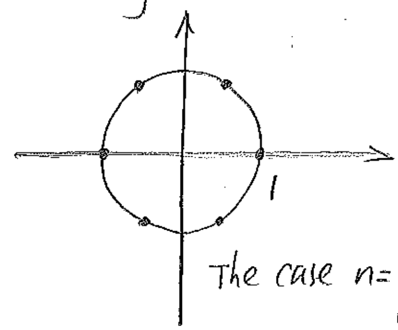
This implies $\begin{cases} r^n = 1, & \text{so } r = 1. \\ n\theta = 2k\pi, & \text{so } \theta = \frac{2k\pi}{n} \end{cases} \Rightarrow z = e^{\frac{2k\pi}{n} i}$ for some $k \in \mathbb{Z}$.

and conversely, if $\theta = \frac{2k\pi}{n}$ for some $k \in \mathbb{Z}$,

$$(e^{i\theta})^n = (e^{\frac{2k\pi i}{n}})^n = e^{2k\pi i} = 1.$$

We conclude the set of n-th roots of unity are
$$\{e^{\frac{2k\pi i}{n}} \in \mathbb{C} \mid 0 \le k \le n-1\}$$

Geometrically, they're evenly distributed on the unit circle



The case $n=6$

More generally, we can follow the same idea:

If $C = Re^{i\theta}$ for some $R > 0$, $\theta \in \mathbb{R}$.

The solutions of $z^n = C$ are

$$\left\{ \sqrt[n]{R}\, e^{\frac{\theta+2k\pi}{n} i} \in \mathbb{C} \;\middle|\; 0 \leq k \leq n-1 \right\}$$

So there're always n solutions to $z^n = C$, where C is a nonzero complex number.
The n solutions are called the n-th roots of C, they're of same norm, and evenly distributed on the circle $|z| = R^{\frac{1}{n}}$.

Example. Solve $z^4 = i$

$i = e^{(\frac{\pi}{2} + 2k\pi)i}$, $k \in \mathbb{Z}$, so $z = e^{\frac{\frac{\pi}{2} + 2k\pi}{4} i}$,

the roots are: $e^{\frac{\pi}{8}i}$, $e^{\frac{5\pi}{8}i}$, $e^{\frac{9\pi}{8}i}$, $e^{\frac{13\pi}{8}i}$