COMPLEX NUMBERS

The concept of complex numbers came from the wish of solving the equation

$$x^2 + 1 = 0.$$

We know this equation has no real number solution since the square of real numbers are all nonnegative.

The way to solve for this equation is to imagine it has a root, and we call it i. (i stands for imaginary)
If i solves $x^2 + 1 = 0$, then i is a "number" such that $i^2 + 1 = 0$.

Since we add the number i to $\mathbb{R}$. we then need to care about how to make the sum of i and real numbers, product of i and real numbers meaningful.

We define a complex number to be a number of the form $z = x + yi$, where x and y are real numbers.

Given two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, we define their sum to be

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2) i$$

and their product to be

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$$

The set of all the complex numbers with the above two operations form a field, called the Field of Complex Numbers, and denoted by $\mathbb{C}$.

We can define subtraction & division of complex numbers based on the definition of addition and multiplication:

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$$

If $z_2 \neq 0$,
$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)}$$
$$= \frac{(x_1 x_2 + y_1 y_2) + (x_2 y_1 - x_1 y_2)i}{x_2^2 + y_2^2}$$
$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} i$$

In particular, given a nonzero complex number $z = x + yi$, its multiplicative inverse is $z^{-1} = \frac{1}{z} = \frac{1}{x+yi} = \frac{x - yi}{x^2 + y^2}$

So this indicates we can understand $\frac{z_1}{z_2}$ as $z_1 \cdot z_2^{-1}$.

Example. $(3+2i) + (4-3i) = (3+4) + (2-3)i = 7 - i$

$$(3+2i)(4-3i) = (3\times4 + 2\times3) + (2\times4 - 3\times3)i$$
$$= 18 - i$$

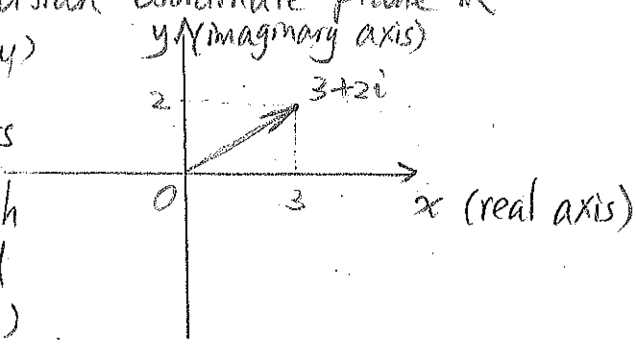
$$\frac{3+2i}{4-3i} = \frac{(3+2i)(4+3i)}{(4-3i)(4+3i)} = \frac{6 + 17i}{4^2 + 3^2} = \frac{6}{25} + \frac{17}{25}i$$

$$(4-3i)^{-1} = \frac{1}{4-3i} = \frac{4+3i}{4^2+3^2} = \frac{4}{25} + \frac{3}{25}i$$

Geometric Presentation of Complex Numbers:

Each complex number $x + yi$ can be identified with the point (x, y) on the Cartisian coordinate plane $\mathbb{R}^2$

Also recall that (x, y) is identified with its "position vector", which starts at $(0, 0)$ and terminates at (x, y).



Given a complex number $z = x + yi$, we call x the real part of z and y the imaginary part of z, denote by

$x = Re(z)$; $y = Im(z)$

We see z is identified with the vector whose first entry is $Re(z)$ and second entry is $Im(z)$.

Recall that given a vector $\vec{v} = (x, y)$, its length is defined as $|\vec{v}| = \sqrt{x^2 + y^2}$.

Using this idea, we find a way to measure the size of a complex number, by the length of its corresponding vector.

Definition. The modulus (or absolute value) of a complex number $z = x + yi$ is $|z| = \sqrt{Re(z)^2 + Im(z)^2} = \sqrt{x^2 + y^2}$

Geometrically, $|z|$ stands for the distance between (x, y) and $(0, 0)$ on the plane, which restricts to the real numbers gives the absolute value of a real number.

Proposition. $|z|^2 = (Re(z))^2 + (Im(z))^2$, $Re(z) \leq |Re(z)| \leq |z|$, $Im(z) \leq |Im(z)| \leq |z|$

Example. Which of $z_1 = -3 + 2i$ and $z_2 = 1 + 4i$ is closer to the origin?

$|z_1|^2 = (-3)^2 + 2^2 = 13$, $|z_2|^2 = 1^2 + 4^2 = 17$.

So $|z_1| = \sqrt{13} < \sqrt{17} = |z_2|$, z_1 is closer to origin

Proposition. (Triangle Inequality). For any complex numbers z_1 and z_2.

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$, and equality holds if and only if z_1, z_2 are on a same ray starting from O.

(ii) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$, and equality holds if and only if z_1, z_2 are on a same ray starting from O.

Proof. These follows directly from the triangle inequality of vectors.

Remark. We can extend the triangle inequality to n complex numbers $z_1, z_2, \ldots, z_n$:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

Proposition. For any $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1| \cdot |z_2|$

Proof. let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ then $|z_1|^2 = x_1^2 + y_1^2$, $|z_2|^2 = x_2^2 + y_2^2$

$$\begin{aligned} |z_1 z_2|^2 = |(x_1 + y_1 i)(x_2 + y_2 i)|^2 &= |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i|^2 \\ &= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 \\ &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) = |z_1|^2 \cdot |z_2|^2 \end{aligned}$$

Definition. $z = x + yi$ is a complex number, we define its complex conjugate to be $\bar{z} = x - yi$

Properties: (i) If $z \in \mathbb{R}$, then $z = \bar{z}$

(ii) $z \cdot \bar{z} = |z|^2$

(iii) $|z| = |\bar{z}|$

(iv) On the complex plane, z and $\bar{z}$ are symmetric along the real axis

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(vi) $\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{z_1 \cdot \bar{z}_2}{|z_2|^2}$

(vii) $\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$, $\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$