Definition. A contour is an arc consisting of a finite number of smooth arcs joined end to end.

Theorem (Jordan Curve Theorem). Every simple closed contour divides the plane into two distinct domains, one of which is bounded (called the interior) while the other is unbounded (called the exterior)

Now we can define contour integrals of complex functions

Definition. If C is a contour represented by $z(t)$, $a \le t \le b$, and f is a complex function such that $f[z(t)]$ is piecewise continuous on $[a,b]$, then define the contour integral of f along C in terms of parameter t to be:

$$\int_C f(z)\,dz = \int_a^b f[z(t)] \cdot z'(t)\,dt$$

It can be varified by the change of variable formula that $\int_C f(z)\,dz$ is independent of the parameterization of C.

Remark. We can form a Riemann Sum Type definition of $\int_C f(z)\,dz$, which will be more intrinsic conceptually. Our definition is more practical here, since it tells us directly how to make the computation.

Definition.
- If C is a contour $z(t)$, $a \le t \le b$, then $-C$ is its opposite contour $z(-t)$, $-b \le t \le -a$, i.e. reverse the order of the points on C.
- If C_1 is a contour from z_1 to z_2, C_2 is a contour from z_2 to z_3, then $C_1 + C_2$ is the contour first going along C_1 and then going along C_2. If C_1 and C_2 have the same final point, we can define $C_1 - C_2 = C_1 + (-C_2)$

Proposition. (i) $\int_{-C} f\,dz = -\int_C f\,dz$

(ii) $\int_{C_1+C_2} f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$

(iii) $\int_{C_1-C_2} f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz$

(iv) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$

(v) $\int_C f(z)+g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

Proof. (i) $\int_{-C} f\,dz = \int_{-b}^{-a} f(z(-t))(-z'(-t))\cdot dt$) change of variable $t=-\tau$

$= -\int_b^a f(z(\tau))\cdot(-z'(\tau))\,d\tau$

$= -\int_a^b f(z(\tau))\cdot z'(\tau)\,d\tau$

$= -\int_C f\,dz$

(ii) If C_1 is parameterized by $z_1(t)$, $a\le t\le b$, we can find a parameterization of C_2 such that $z_2(t)$, $b\le t\le c$.

So $\int_{C_1+C_2} f(z)\,dz = \int_a^c f(z(t))z'(t)\,dt = \int_a^b f(z_1(t))z_1'(t)\,dt + \int_b^c f(z_2(t))z_2'(t)\,dt$

$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

The proofs for (iii), (iv), (v) are left as exercises

Example. Let C be the unit circle path $z(t)=e^{it}$, $0\le t\le 2\pi$

$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\cdot(e^{it})'\,dt = \int_0^{2\pi} i\cdot dt = 2\pi i$

Example. C is a contour consisting of finitely many arcs in the order of $C_1, C_2, \ldots, C_n$, with a set of $n+1$ ordered endpoints $z_0, z_1, z_2, \ldots, z_n$ such that C_k begins at z_{k-1} and ends at z_k. Consider the integral

$$\int_C z\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz + \cdots + \int_{C_n} z\,dz$$

For C_k parameterized by $z(t)$, $a \le t \le b$ (so $z(a) = z_{k-1}$, $z(b) = z_k$)

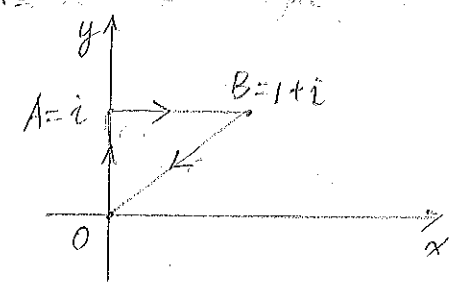
$$\int_{C_k} z\,dz = \int_a^b z(t)\cdot z'(t)\,dt$$

Note that $([z(t)]^2)' = 2z(t)\,z'(t)$ so $\left(\frac{z(t)^2}{2}\right)' = z(t)z'(t)$ by the generalized Fundamental Theorem of Calculus.

$$\int_{C_k} z\,dz = \int_a^b z(t)\,z'(t)\,dt = \frac{z(b)^2}{2} - \frac{z(a)^2}{2} = \frac{z_k^2}{2} - \frac{z_{k-1}^2}{2}$$

So $\displaystyle\int_C z\,dz = \sum_{k=1}^{n} \frac{z_k^2}{2} - \frac{z_{k-1}^2}{2} = \frac{z_n^2}{2} - \frac{z_0^2}{2}$

Example. Let C be the path along the triangle OAB in clockwise direction, starting and ending at O.



Then for a given function $f(z) = (y-x) - 3x^2 i$. $(z = x+iy)$

$$\int_C f(z)\,dz = \int_{\overline{OA}} f(z)\,dz + \int_{\overline{AB}} f(z)\,dz - \int_{\overline{OB}} f(z)\,dz$$

$\overline{OA}$ can be parameterized by $z_1(y) = yi$, $0 \le y \le 1$.

so $\displaystyle\int_{\overline{OA}} f(z)\,dz = \int_0^1 f(yi)\cdot z_1'(y)\,dy = \int_0^1 y\cdot i\,dy = \frac{1}{2}i$

$\overline{AB}$ can be parameterized by $z_2(x) = x + i$, $0 \le x \le 1$

$$\int_{\overline{AB}} f(z)\,dz = \int_0^1 f(x+i)\, z_2'(x)\,dx = \int_0^1 [(1-x) - 3x^2 i]\cdot 1\,dx$$
$$= \frac{1}{2} - i$$

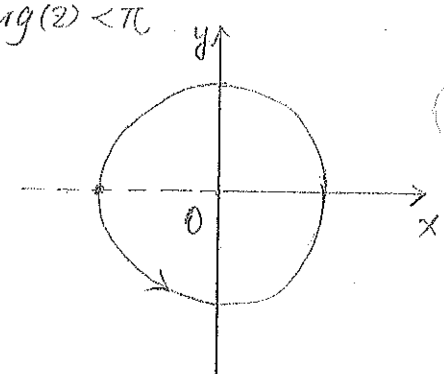
$\overline{OB}$ can be parameterized by $z_3(t) = t + ti$, $0 \le t \le 1$

$$\int_{\overline{OB}} f(z)\,dz = \int_0^1 f(t+ti)\, z'(t)\,dt = \int_0^1 -3t^2 i\cdot(1+i)\,dt = 1 - i$$

So $\displaystyle\int_C f(z)\,dz = \int_{\overline{OA}} f(z)\,dz + \int_{\overline{AB}} f(z)\,dz - \int_{\overline{OB}} f(z)\,dz = \frac{i}{2} + \left(\frac{1}{2} - i\right) - (1-i) = -\frac{1}{2} + \frac{1}{2}i$

Example. Let C be the circle path $z(t) = e^{it}$, $-\pi \le t \le \pi$.

$f(z) = \log(z)$ is the branch $-\pi < \arg(z) < \pi$



Though the branch is not defined on the branch cut, removing one point from the integral doesn't affect the integral, so we get

$$\int_C f(z)\,dz = \lim_{\varepsilon\to 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon} \log(e^{it})\cdot(e^{it})'\,dt$$

$$= \lim_{\varepsilon\to 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon} (it)\cdot i e^{it}\,dt$$

$$= \lim_{\varepsilon\to 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon} -t\cos t - it\sin t\,dt$$

$$= \lim_{\varepsilon\to 0} (-t\sin t - \cos t) - i(\sin t - t\cos t)\Big|_{-\pi+\varepsilon}^{\pi-\varepsilon}$$

$$= \lim_{\varepsilon\to 0} -(\cos t + i\sin t) + i\cdot t(\cos t + i\sin t)\Big|_{-\pi+\varepsilon}^{\pi-\varepsilon}$$

$$= \lim_{\varepsilon\to 0} -e^{it} + it\,e^{it}\Big|_{-\pi+\varepsilon}^{\pi-\varepsilon}$$

$$= -2\pi i.$$