DEFINITE INTEGRAL

We are going to consider a function \( W : I \to \mathbb{C} \) given by \( W(t) = u(t) + v(t) i \), where \( I = [a, b] \) is some interval on \( \mathbb{R} \), and \( u(t), v(t) \) are real-valued functions on \( I \).

**Definition.** If the derivatives \( u(t) \) and \( v(t) \) exist, define the derivative \( W'(t) = u'(t) + v(t)i \).

**Example.** \( W(t) = e^{zot} \), where \( z_0 = x_0 + y_0i \) is a constant.

\[
W(t) = e^{z_0 t} = e^{x_0 t} \cdot e^{y_0 t} = e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t
\]

So \( W'(t) = (e^{x_0 t} \cos y_0 t)' + (i e^{x_0 t} \sin y_0 t)' \)

\[
= x_0 e^{x_0 t} \cos y_0 t - y_0 e^{x_0 t} \sin y_0 t + x_0 i e^{x_0 t} \sin y_0 t + y_0 i e^{x_0 t} \cos y_0 t
\]

\[
= z_0 e^{x_0 t} \cos y_0 t + z_0 i e^{x_0 t} \sin y_0 t
\]

\[
= z_0 e^{x_0 t} \cdot e^{y_0 t}
\]

\[
= z_0 e^{z_0 t}
\]

**Proposition.** If \( W : I \to \mathbb{C} \) and \( f : \mathbb{C} \to \mathbb{C} \), if \( \frac{d}{dt} (w(t)) \) and \( \frac{d^2}{dt^2} (w(t)) \) both exist, then \( \frac{d}{dt} (w(t)) \) exists and \( \frac{d^2}{dt^2} (w(t)) = \frac{df}{dz} (w(t)) \cdot \frac{dw}{dt} (t) \).

Write \( W(t) = u(t) + v(t) i \),

\[
f(z) = f(x + yi) = U(x, y) + V(x, y) i
\]

Then \( f \circ W(t) = U(u(t), v(t)) + V(u(t), v(t)) i \).
\[ \frac{d}{dt} (f \cdot w) = \frac{d}{dt} \sum (u(t), v(t)) + i \cdot \frac{d}{dt} \sum (w(t), v(t)) \]

\[ = \frac{\partial u}{\partial x} \cdot \frac{du}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dv}{dt} + i \frac{\partial v}{\partial x} \cdot \frac{du}{dt} + i \frac{\partial v}{\partial y} \cdot \frac{dv}{dt} \]

\[ = (\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}) \frac{du}{dt} + (\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}) \frac{dv}{dt} \]

\[ = \frac{df}{dz} \cdot \frac{du}{dt} + i \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} i \right) \frac{dv}{dt} \]

\[ = \frac{df}{dz} \left( \frac{du}{dt} + i \frac{dv}{dt} \right) \]

\[ = \frac{df}{dz} \cdot \frac{dw}{dt} \]

**Example.** If \( W(t) : J \rightarrow C \) has derivative, then \( \frac{d}{dt} W(t)^2 = 2W(t) \cdot W(t) \)

**Definition.** If \( w(t) = u(t) + vi(t) \), define the definite integral of \( w(t) \) over an interval \( [a, b] \) to be

\[ \int_a^b w(t) \, dt = \int_a^b u(t) \, dt + i \int_a^b v(t) \, dt \]

provided the integrals \( \int_a^b u(t) \, dt \), \( \int_a^b v(t) \, dt \) exist.

**Example.**

\[ \int_0^\pi e^{it} \, dt = \int_0^\pi (\cos t + i \sin t) \, dt = \int_0^\pi \cos t \, dt + i \int_0^\pi \sin t \, dt \]

\[ = \frac{\pi}{2} + i (1 - \frac{\pi}{2}) \]

**Example.** The Fundamental Theorem of Calculus can also be extended to this definition:

If \( W(t) = w(t) \), then \( \int_a^b w(t) \, dt = W(b) - W(a) \)
Its proof is straightforward, just consider each of the real part and the imaginary part:

If \( W(t) = U(t) + V(t)i \),
\[
W(t) = u(t) + v(t)i,
\]
and \( W'(t) = U'(t) + V'(t)i = w(t) \).

We see \( U'(t) = u(t) \), \( V'(t) = v(t) \),
\[
\int_{a}^{b} w(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt,
\]
\[
= U(b) - U(a) + i(V(b) - V(a))
\]
\[
= W(b) - W(a).
\]

**Proposition.** If \( w(t) = u(t) + iv(t) \), and \( u(t) \), \( v(t) \) are piecewise continuous on \([a, b] \), then \( \int_{a}^{b} w(t) \, dt \) exists.
Definition. An arc in the complex plane is a continuous function \( [a, b] \rightarrow \mathbb{C} \) \( z(t) = x(t) + y(t)i \), with image points ordered according to increasing values of \( t \).

Example. \( z(t) = t + t^2i \), \( t \in [0, 1] \) is a part of a parabola.

Definition. An arc \( C \) is simple if it has no self-intersection, i.e. \( t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2) \).

An arc \( C \) is closed if \( z(a) = z(b) \).

\( C \) is a simple closed curve, or a Jordan curve, if it's simple and closed. It's positively oriented if it's in counterclockwise direction.

Example. The unit circle \( z = e^{i\theta}, 0 \leq \theta \leq \pi \) is a simple closed curve, positively oriented.

\( z = e^{i\theta}, 0 \leq \theta \leq 2\pi \) is a simple closed curve, negatively oriented.

Note. Arcs are functions, so they may still be different arcs even if their images are the same geometric figure.

For example. \( z = e^{i2\theta} (0 \leq \theta \leq 2\pi) \) is a different arc from \( z = e^{i\theta} (0 \leq \theta \leq 2\pi) \).
But at the same time, the same arc may admit different parameterizations.

If $Z(t)$ is a curve defined on $t \in [a, b]$ and $t = \phi(t)$ is a strictly increasing continuously differentiable on $[\alpha, \beta]$ with $\phi(\alpha) = a$, $\phi(\beta) = b$, then $Z(t) = Z(\phi(t)) \alpha \leq t \leq \beta$ represents the same arc as $Z(t)$, $a \leq t \leq b$.

Example $Z = e^{it}, 0 \leq \theta \leq \pi$ and $Z = e^{i\omega}, 0 \leq \omega \leq \pi$ represent the same arc.

Definition. If an arc is given by $Z(t) = x(t) + y(t)i$, we say $Z(t)$ is differentiable if $Z'(t)$ exists. Recall that the arc length of an arc on $\mathbb{R}^2$ is defined to be $\int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, dt$. We see the arc length of the arc $Z(t)$, $a \leq t \leq b$ is computed by $\int_a^b |Z'(t)| \, dt$.

Exercise. Prove the arc length doesn't depend on parameterization.

Definition. If $C$ is a differentiable arc parameterized by $Z(t)$ with $Z'(t) \neq 0$ on $(a, b)$, define the unit tangent vector of $C$ at $Z(t)$ to be $\hat{T} = \frac{Z'(t)}{|Z'(t)|}$.

Definition. An arc is smooth if $Z'(t)$ is a continuous function.