DEFINITE INTEGRAL

We are going to consider a function $w: I \longrightarrow \mathbb{C}$ given by $w(t) = u(t) + v(t) i$, where $I = [a, b]$ is some interval on $\mathbb{R}$, and $u(t)$, $v(t)$ are real-valued functions on I

Definition. If the derivatives $u'(t)$ and $v'(t)$ exist, define the derivative $w'(t) = u'(t) + v'(t) i$

Example. $w(t) = e^{z_0 t}$, where $z_0 = x_0 + y_0 i$ is a constant.

$$w(t) = e^{z_0 t} = e^{x_0 t} \cdot e^{i y_0 t}$$
$$= e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t$$

So $w'(t) = (e^{x_0 t} \cos y_0 t)' + (i e^{x_0 t} \sin y_0 t)'$

$$= x_0 e^{x_0 t} \cos y_0 t - y_0 e^{x_0 t} \sin y_0 t + x_0 i e^{x_0 t} \sin y_0 t + y_0 i e^{x_0 t} \cos y_0 t$$
$$= z_0 e^{x_0 t} \cos y_0 t + z_0 i e^{x_0 t} \sin y_0 t$$
$$= z_0 e^{x_0 t} \cdot e^{i y_0 t}$$
$$= z_0 e^{(x_0 + i y_0) t}$$
$$= z_0 e^{z_0 t}$$

Proposition. $w: I \longrightarrow \mathbb{C}$ and $f: \mathbb{C} \longrightarrow \mathbb{C}$, If $\frac{dw}{dt}(t)$ and $\frac{df}{\partial z}(w(t))$ both exist, then $\frac{d f\circ w}{dt}(t)$ exists and $\frac{d f\circ w}{dt}(t) = \frac{df}{dz}(w(t)) \cdot \frac{dw}{dt}(t)$

Write $w(t) = u(t) + v(t) i$,

$$f(z) = f(x + yi) = U(x, y) + V(x, y) i$$

Then $f \circ w(t) = U(u(t), v(t)) + V(u(t), v(t)) i$

$$\frac{d}{dt}(f\circ w) = \frac{d}{dt}U(u(t), v(t)) + i\cdot\frac{d}{dt}V(u(t), v(t))$$

$$= \frac{\partial U}{\partial x}\cdot\frac{du}{dt} + \frac{\partial U}{\partial y}\cdot\frac{dv}{dt} + i\,\frac{\partial V}{\partial x}\cdot\frac{du}{dt} + i\cdot\frac{\partial V}{\partial y}\cdot\frac{dv}{dt}$$

$$= \left(\frac{\partial U}{\partial x} + i\cdot\frac{\partial V}{\partial x}\right)\frac{du}{dt} + \left(\frac{\partial U}{\partial y} + i\frac{\partial V}{\partial y}\right)\frac{dv}{dt}$$

$$= \frac{df}{dz}\cdot\frac{du}{dt} + i\left(\frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}\,i\right)\frac{dv}{dt}$$

$$= \frac{df}{dz}\,\frac{du}{dt} + i\cdot\frac{df}{\partial z}\,\frac{dv}{dt}$$

$$= \frac{df}{dz}\left(\frac{du}{dt} + i\frac{dv}{dt}\right)$$

$$= \frac{df}{dz}\cdot\frac{dw}{dt}$$

Example. If $w(t) : I \to \mathbb{C}$ has derivative, then $\frac{d}{dt}w(t)^2 = 2w(t)\cdot w'(t)$

Definition. If $w(t) = u(t) + v(t)i$, define the definite integral of $w(t)$ over an interval $[a, b]$ to be

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

provided the integrals $\int_a^b u(t)\,dt$, $\int_a^b v(t)\,dt$ exist.

Example. $$\int_0^{\frac{\pi}{4}} e^{it}\,dt = \int_0^{\frac{\pi}{4}}(\cos t + i\sin t)\,dt = \int_0^{\frac{\pi}{4}}\cos t\,dt + i\int_0^{\frac{\pi}{4}}\sin t\,dt$$

$$= \frac{\sqrt{2}}{2} + i\left(1 - \frac{\sqrt{2}}{2}\right)$$

Example. The Fundamental Theorem of Calculus can also be extended to this definition:

If $W'(t) = w(t)$, then $\int_a^b w(t)\,dt = W(b) - W(a)$

Its proof is straight forward, just consider each of the real part and the imaginary part:

If $W(t) = U(t) + V(t)\,i$

$w(t) = u(t) + v(t)\,i$,

and $W'(t) = U'(t) + V'(t)\,i = w(t)$

we see $U'(t) = u(t)$, $V'(t) = v(t)$

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$$= U(b) - U(a) + i\,(V(b) - V(a))$$

$$= W(b) - W(a)$$

Proposition. If $w(t) = u(t) + iv(t)$, and $u(t)$, $v(t)$ are piecewise continuous on $[a,b]$, then $\int_a^b w(t)\,dt$ exists.

CONTOUR INTEGRALS.

Definition. An arc in the complex plane is a continuous function $[a,b] \longrightarrow \mathbb{C}$ $z(t) = x(t) + y(t)i$, with image points ordered according to increasing values of t.

Example. $z(t) = t + t^2 i$, $t \in [0,1]$ is a part of a parabola.

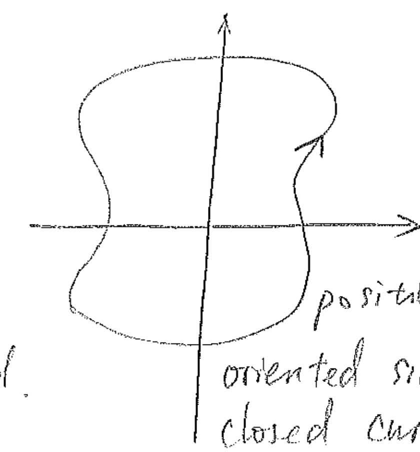
Definition. An arc C is simple if it has no self-intersection. i.e. $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$

An arc C is closed if $z(a) = z(b)$.

C is a simple closed curve, or a Jordan curve if it's simple and closed. It's positively oriented if it's in counterclockwise direction.

Example The unit circle $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ is a simple closed curve, positively oriented.

$z = e^{-i\theta}$, $0 \le \theta \le 2\pi$ is a simple closed curve, negatively oriented.



positively oriented simple closed curve

Note: Arcs are functions, so they may still be different arcs even if their images are the same geometric figure.

For example $z = e^{i2\theta}$ $(0 \le \theta \le 2\pi)$ is a different arc from $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$

But at the same time, the same arc may admit different parametrizations:

If $z(t)$ is a curve defined on $t \in [a, b]$ and $t = \phi(\tau)$ is a strictly increasing continuously differentiable on $[\alpha, \beta]$ with $\phi(\alpha) = a$, $\phi(\beta) = b$ then $Z(\tau) = z(\phi(\tau))$ $\alpha \le \tau \le \beta$ represents the same arc as $z(t)$, $a \le t \le b$.

Example $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ and $z = e^{i2\alpha}$, $0 \le \alpha \le \pi$ represent the same arc.

Definition. If an arc is given by $z(t) = x(t) + y(t)i$, we say $z(t)$ is differentiable if $z'(t)$ exists.

Recall that the arclength of an arc on $\mathbb{R}^2$ is defined to be $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$, we see the arclength of the arc $z(t)$ $a \le t \le b$ is computed by $\int_a^b |z'(t)|\, dt$.

Exercise. Prove the arc length doesn't depend on parameterization.

Definition. If C is a differentiable arc parameterized by $z(t)$ with $z'(t) \ne 0$ on (a, b) define the unit tangent vector of C at $z(t)$ to be $\hat{T} = \dfrac{z'(t)}{|z'(t)|}$

Definition. An arc is smooth if $z'(t)$ is a continuous function.