THE TRIGONOMETRIC FUNCTIONS

We have defined $e^{ix} = \cos x + i\sin x$ for any real $x \in \mathbb{R}$

Replace x by $-x$, we get $e^{-ix} = \cos(-x) + i\sin(-x)$
$= \cos x - i\sin x$

So we have a system of two equations

$$\begin{cases} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{cases}$$

which implies $\begin{cases} \cos x = \dfrac{e^{ix}+e^{-ix}}{2} \\ \sin x = \dfrac{e^{ix}-e^{-ix}}{2i} \end{cases}$

Since we have extended the exponential function to $\mathbb{C}$, it's natural now to define the sine and cosine functions:

$$\begin{cases} \sin z = \dfrac{e^{iz}-e^{-iz}}{2i} \\ \cos z = \dfrac{e^{iz}+e^{-iz}}{2} \end{cases} \qquad (z \in \mathbb{C})$$

Example $\sin \pi i = \dfrac{e^{i(\pi i)} - e^{-i(\pi i)}}{2i} = \dfrac{e^{-\pi} - e^{\pi}}{2i}$

$\cos \pi i = \dfrac{e^{i(\pi i)} + e^{-i(\pi i)}}{2} = \dfrac{e^{-1} + e}{2}$

Proposition (i) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

(ii) $\sin^2 z + \cos^2 z = 1$.

The proofs are left as exercises.

Proposition $\sin z$ and $\cos z$ are entire functions, with

$(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$

Proof. They're entire since they're linear combinations of the entire functions e^{iz} & e^{-iz}

Then apply the chain rule to find their derivatives

Proposition. $\sin z$ and $\cos z$ both have period 2π

Proof. $$\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{iz}\cdot e^{2\pi i} - e^{-iz}\cdot e^{-2\pi i}}{2i} = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos(z+2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{iz}\cdot e^{2\pi i} + e^{-iz}\cdot e^{-2\pi i}}{2} = \frac{e^{iz}+e^{-iz}}{2}$$

Remark. $\sin z$ and $\cos z$ are NOT bounded, which is different from the real case.

For example, $\cos(Ni) = \dfrac{e^{i(Ni)} + e^{-i(Ni)}}{2} = \dfrac{e^{-N}+e^{N}}{2}$

we see $\lim\limits_{N\to+\infty} \cos(Ni) = \infty$

Definition A zero of a function f is a number z_0 such that $f(z_0)=0$

Example The real function $f(x) = x^2+1$ has no zeros
The complex function $f(z) = z^2+1$ has $\pm i$ as zeros

Proposition. All the zeros of $\sin z$ and $\cos z$ are real.

Proof $$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{1}{2i}\left[e^{ix}\cdot e^{-y} - e^{-ix}\cdot e^{y}\right]$$

We see $\sin z = 0$ iff. $e^{ix} \cdot e^{-y} = e^{-ix} \cdot e^{y}$

iff. $e^{2ix} = e^{2y}$

Note $|e^{i \cdot 2x}| = 1$, so $e^{2ix} = e^{2y} \Rightarrow e^{2y} = 1 \Rightarrow y = 0$

which further implies $e^{i \cdot 2x} = 1 \Rightarrow 2x = 2k\pi, k \in \mathbb{Z}$.

$\Rightarrow x = k\pi, k \in \mathbb{Z}$

We see the only zeros of $\sin z$ are $k\pi, k \in \mathbb{Z}$.

Similarly, we can prove that the zeros of $\cos z$ are $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

Definition. $\tan z = \dfrac{\sin z}{\cos z}$ $\quad \cot z = \dfrac{\cos z}{\sin z}$

$\sec z = \dfrac{1}{\cos z}$ $\quad \csc z = \dfrac{1}{\sin z}$

Definition. If f is not analytic at z_0, but is analytic at some point in every neighbourhood of z_0, then z_0 is called a singular point, or singularity of f.

By this definition, we see the zeros of $\cos z$ are the singularities of $\tan z$ and $\sec z$, and the zeros of $\sin z$ are the singularities of $\cot z$ and $\csc z$