THE TRIGONOMETRIC FUNCTIONS

We have defined $e^{ix} = \cos x + i\sin x$ for any real $x \in \mathbb{R}$.

Replace $x$ by $-x$, we get $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$

So we have a system of two equations

\[
\begin{align*}
    e^{ix} &= \cos x + i\sin x \\
    e^{-ix} &= \cos x - i\sin x
\end{align*}
\]

which implies

\[
\begin{align*}
    \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
    \sin x &= \frac{e^{ix} - e^{-ix}}{2i}
\end{align*}
\]

Since we have extended the exponential function to $\mathbb{C}$, it's natural now to define the sine and cosine functions:

\[
\begin{align*}
    \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\
    \cos z &= \frac{e^{iz} + e^{-iz}}{2}
\end{align*}
\] (\(z \in \mathbb{C}\))

Example

\[
\begin{align*}
    \sin \pi i &= \frac{e^{i\pi i} - e^{-i\pi i}}{2i} = \frac{e^{\pi} - e^{-\pi}}{2i} \\
    \cos \pi i &= \frac{e^{i\pi i} + e^{-i\pi i}}{2} = \frac{e^1 + e^{-1}}{2}
\end{align*}
\]

Proposition

(i) $\sin(-z) = -\sin z$, \quad $\cos(-z) = \cos z$

(ii) $\sin^2 z + \cos^2 z = 1$

The proofs are left as exercises.
Proposition. \( \sin z \) and \( \cos z \) are entire functions, with 
\[
(\sin z)' = \cos z \quad \text{and} \quad (\cos z)' = -\sin z
\]

Proof. They're entire since they're linear combinations of the 
entire functions \( e^{iz} \) & \( e^{-iz} \).

Then apply the chain rule to find their derivatives.

Proposition. \( \sin z \) and \( \cos z \) both have period \( 2\pi \).

Proof. 
\[
\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{iz}e^{2\pi i} - e^{-iz}e^{2\pi i}}{2i} = \frac{e^{iz} - e^{-iz}}{2i}
\]
\[
\cos(z+2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{iz}e^{2\pi i} + e^{-iz}e^{2\pi i}}{2} = \frac{e^{iz} + e^{-iz}}{2}
\]

Remark. \( \sin z \) and \( \cos z \) are NOT bounded, which is different 
from the real case.

For example, \( \cos(Ni) = \frac{e^{i(Ni)} + e^{-i(Ni)}}{2} = \frac{e^N + e^{-N}}{2} \)

We see \( \lim_{N \to +\infty} \cos(Ni) = \infty \)

Definition. A zero of a function \( f \) is a number \( z_0 \) such that \( f(z_0) = 0 \)

Example. The real function \( f(x) = x^2 + 1 \) has no zeros.
The complex function \( f(z) = z^2 + 1 \) has \( \pm i \) as zeros.

Proposition. All the zeros of \( \sin z \) and \( \cos z \) are real.

Proof. 
\[
\sin z = \frac{e^{i(z+iy)} - e^{-i(z+iy)}}{2i}
\]
\[
= \frac{1}{2i} \left[ e^{ix}e^{iy} - e^{-ix}e^{-iy} \right]
\]
We see $\sin z = 0$ iff $e^{ix} \cdot e^{-y} = e^{-ix} \cdot e^y$

iff $e^{2ix} = e^{2y}$

Note $|e^{i \cdot 2\alpha}| = 1$, so $e^{2ix} = e^{2y} \Rightarrow e^{2y} = 1 \Rightarrow y = 0$

which further implies $e^{i \cdot 2\alpha} = 1 \Rightarrow 2\alpha = 2k\pi, k \in \mathbb{Z}$

$\Rightarrow \alpha = k\pi, k \in \mathbb{Z}$

We see the only zeros of $\sin z$ are $k\pi, k \in \mathbb{Z}$.

Similarly, we can prove that the zeros of $\cos z$ are $\frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

**Definition.**

$tan z = \frac{\sin z}{\cos z}$ \quad $\cot z = \frac{\cos z}{\sin z}$

$\sec z = \frac{1}{\cos z}$ \quad $\csc z = \frac{1}{\sin z}$

**Definition.** If $f$ is not analytic at $z_0$, but is analytic at some point in every neighborhood of $z_0$, then $z_0$ is called a singular point or singularity of $f$.

By this definition, we see the zeros of $\cos z$ are the singularities of $\tan z$ and $\sec z$, and the zeros of $\sin z$ are the singularities of $\cot z$ and $\csc z$.