Example. Verify $\mathrm{Log}[(1+i)^2] = 2\,\mathrm{Log}(1+i)$

$$\mathrm{Log}[(1+i)^2] = \mathrm{Log}(2i) = \ln|2i| + i\,\mathrm{Arg}(2i) = \ln 2 + \frac{\pi}{2} i$$

$$2\,\mathrm{Log}(1+i) = 2\left(\ln|1+i| + i\,\mathrm{Arg}(1+i)\right) = 2\left(\ln\sqrt{2} + \frac{\pi}{4} i\right)$$
$$= \ln 2 + \frac{\pi}{2} i$$

Example. Verify $\log(i^2) \neq 2\log i$

$$\log i^2 = \log(-1) = \ln|-1| + i\arg(-1) = (\pi + 2k\pi)i, \ k \in \mathbb{Z}.$$

$$2\log i = 2\left(\ln|i| + i\arg(i)\right) = 2\,i\left(\frac{\pi}{2} + 2k\pi\right) = (\pi + 4k\pi)i, \ k \in \mathbb{Z}$$

Example. $\mathrm{Log}(-1)^2 = \mathrm{Log}\,1 = 0, \quad 2\,\mathrm{Log}(-1) = 2(i\pi) = 2\pi i$

So $\mathrm{Log}(-1)^2 \neq 2\,\mathrm{Log}(-1)$

We now want one more step further: we wish to have a single value logarithmic function that is continuous:

Let $\alpha \in \mathbb{R}$, we delete the ray $\theta = \alpha$ from the domain, and define

$$\log(z) = \ln|z| + i\theta, \quad \text{where } \theta = \arg(z) \cap (\alpha, \alpha + 2\pi)$$

Then the function we obtain is continuous on this smaller domain since both real and imaginary parts are continuous now.

In your homework, you have checked the polar form of the Cauchy-Riemann Equations are $\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases}$

$\log(re^{i\theta}) = \ln r + i\theta$, so $\begin{cases} r u_r = r \cdot \frac{1}{r} = 1 = v_\theta \\ u_\theta = 0 = -r \cdot 0 = -r v_r \end{cases}$

So this function is analytic.

Next, we are going to use the expression of $f'(z)$ in the polar form: $f'(z) = e^{-i\theta}(u_r + i v_r)$

So for $f(z) = \log z$,

$$f'(z) = e^{-i\theta}\left(\frac{\partial \ln r}{\partial r} + r\,\frac{\partial \theta}{\partial r}\right)$$

$$= e^{-i\theta} \cdot \frac{1}{r}$$

$$= \frac{1}{re^{i\theta}}$$

$$= \frac{1}{z}$$

we thus get : $(\log z)' = \frac{1}{z}$

Definition: A branch of a multi-value function f is a single-value function F that is analytic in some domain at each z of which $F(z)$ is one of the values of $f(z)$.

When applied to $\log z$, each $\log z = \ln|z| + i\theta$. $(\alpha < \theta < \alpha + 2\pi)$ is a branch, and the function $\operatorname{Log} z = \ln|z| + i\theta$. $(-\pi < \theta < \pi)$ is called the principal branch.

Definition:
- A branch cut is a portion of a line or curve that is removed from $\mathbb{C}$ in order to define a branch F of a multi-value function f.
- Points on the branch cut are called singular points for F.
- A point that is common to all branch cuts of f is called a branch point

Example: For $\operatorname{Log} z = \ln|z| + i\theta$ $(-\pi < \theta < \pi)$ the ray $\theta = \pi$ is the branch cut, and 0 is a branch point for the multi-value function $\log z$

Remark. We can also obtain the derivative of a branch $f(z) = \log z = \ln r + i\theta$, $(\alpha < \theta < \alpha + 2\pi)$ by the Chain Rule.

First, $e^{\log z} = z$, so we can differentiate both sides using the Chain Rule.

$$\frac{d}{dz}(e^{\log z}) = \frac{d}{dz}(z)$$

$$e^{\log z} \cdot (\log z)' = 1$$

$$z \cdot (\log z)' = 1$$

$$\log z = \frac{1}{z}$$

By making a branch cut, we can obtain an analytic branch of the $\log z$ multi-valued function, but there're also properties that are only true when we regard $\log z$ as a multi-valued function.

Example. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

This is because

$$\begin{aligned}\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2)\\ &= \ln|z_1| + \ln|z_2| + i(\arg(z_1) + \arg(z_2))\\ &= \ln|z_1| + i\arg(z_1) + \ln|z_2| + i\arg(z_2)\\ &= \log(z_1) + \log(z_2)\end{aligned}$$

(Note $\arg(z)$ is a set instead of a number!)

But if we take a branch cut, $\log(z) = \ln|z| + i\theta$, $(0 < \theta < 2\pi)$, then $\log((-i)\cdot(-i)) = \log(-1) = \pi i$, while $\log(-i) + \log(-i) = \frac{3}{2}\pi i + \frac{3}{2}\pi i = 3\pi i$.

THE POWER FUNCTION

We are going to study functions of the form $f(z) = z^c$, where $c \in \mathbb{C}$ is a constant.

First, we know what $f(z) = z^n$ is when $n \in \mathbb{N}$:

$$f(z) = z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n \text{ copies of } z}$$

An important observation which will be useful later is that

$$z^n = e^{n \log z}.$$

This can be verified by:

$$\begin{aligned} e^{n \log z} = e^{n(\ln|z| + i\arg(z))} &= e^{n \ln|z|} \cdot e^{i n \arg(z)} \\ &= e^{\ln|z|^n} \cdot e^{i n \arg(z)} \\ &= |z|^n \cdot e^{i n \arg(z)} \\ &= z^n \end{aligned}$$

Next, let's think about the function $f(z) = z^{\frac{1}{n}}$, $n \in \mathbb{N}$.
Recall that we've discussed about this kind of function before.
If $w = z^{\frac{1}{n}}$, there're n solutions for w when $z \neq 0$.

i.e. $w^n = z = r e^{i\arg(z)}$

so $w = \sqrt[n]{r}\, e^{i \frac{1}{n} \arg(z)}$

Observe
$$\begin{aligned} \tfrac{1}{n} \arg(z) &= \tfrac{1}{n} \{ \mathrm{Arg}(z) + 2k\pi \in \mathbb{R} \mid k \in \mathbb{Z} \} \\ &= \left\{ \frac{\mathrm{Arg}(z)}{n} + \frac{2k\pi}{n} \in \mathbb{R} \;\middle|\; k \in \mathbb{Z} \right\} \\ &= \left\{ \frac{\mathrm{Arg}(z)}{n} + \frac{2k\pi}{n} \in \mathbb{R} \;\middle|\; k \in \mathbb{Z} \cap [0, n-1] \right\} \end{aligned}$$

This implies

$$f(z) = z^{\frac{1}{n}} = \left\{ |z|^{\frac{1}{n}} \cdot e^{i\theta} \in \mathbb{C} \;\middle|\; \theta = \frac{Arg(z)}{n} + \frac{2k\pi}{n},\ 0 \le k \le n-1 \right\}$$

Then observe $z^{\frac{1}{n}} = e^{\frac{1}{n}\log z}$ is also true:

$$e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + i\arg(z))} = e^{\frac{1}{n}\ln|z|} \cdot e^{i\frac{\arg(z)}{n}}$$

$$= \left\{ |z|^{\frac{1}{n}} \cdot e^{i\theta} \in \mathbb{C} \;\middle|\; \theta = \frac{Arg(z)+2k\pi}{n},\ 0 \le k \le n-1 \right\}$$

$$= z^{\frac{1}{n}}$$

The above observations motivates the definition:

Definition | If $c \in \mathbb{C}$, the power function $f(z) = z^c$ is defined by

$$z^c = e^{c \log z} \quad (z \neq 0)$$

Remark | In general, it'll be a multi-valued function.

Example | $(\sqrt{2})^i = e^{i\log\sqrt{2}} = e^{i(\ln\sqrt{2} + i \cdot 2\pi k)} = e^{-2\pi k + i\ln\sqrt{2}}$, $k \in \mathbb{Z}$.

So there're infinitely many values of $(\sqrt{2})^i$

Proposition | $(z^c)^{-1} = z^{-c}$ (We leave the proof as homework)

Similar to the case of $\log z$, we can make z^c a single valued function by choosing a branch (making a branch cut)
Since $z^c = e^{c\log z}$, once we choose a branch for $\log z$, $\log z$ will be single-valued, therefore z^c will also be single-valued.

For a branch of $\log z$, it's analytic, and exponential function is also analytic, the composition $z^n = e^{c \log z}$ is also analytic on a branch $\alpha < \theta < \alpha + 2\pi$:

$$\frac{d}{dz}(z^c) = \frac{d}{dz} e^{c\log z} = e^{c \log z} \cdot (c \log z)' = e^{c \log z} \cdot \frac{c}{z}$$

$$= c \cdot \frac{e^{c \log z}}{e^{\log z}}$$

$$= c\, e^{(c-1)\log z}$$

$$= c\, z^{c-1}$$

And similar as the $\log(z)$ function, we take $-\pi < \theta < \pi$ as the principal branch of z^c