THE EXPONENTIAL FUNCTIONS

Definition. The exponential function $f(z)=e^z$ is defined by

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

Proposition (i) If $z = x+yi \in \mathbb{C}$, then $|e^z| = e^x$

(ii) $e^{z_1}e^{z_2} = e^{z_1+z_2} \quad \forall\, z_1, z_2 \in \mathbb{C}$

(iii) $f(z)=e^z$ is an entire function, and $f'(z) = e^z$

Proof. (i). $|e^z| = |e^x \cdot e^{iy}| = |e^x|\cdot|e^{iy}| = e^x \cdot 1 = e^x$

(ii) If $z_1 = x_1+y_1 i$, $z_2 = x_2 + y_2 i$,

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}e^{y_1 i}e^{x_2}e^{y_2 i} \\ &= (e^{x_1}e^{x_2})(e^{y_1 i}e^{y_2 i}) \\ &= e^{x_1+x_2}\, e^{(y_1+y_2)i} \\ &= e^{z_1+z_2} \end{aligned}$$

(iii) $f(z) = e^z = e^x\cos y + i e^x \sin y$

So $u(x,y) = e^x\cos y$, $v(x,y) = e^x \sin y$

$$\begin{cases} u_x = e^x\cos y = v_y \\ u_y = -e^x\sin y = -v_x \end{cases}$$

We see u_x, u_y, v_x, v_y are all continuous, satisfying Cauchy-Riemann Equations everywhere, so $f(z)=e^z$ is entire.

$$\begin{aligned} f'(z) &= u_x(x,y) + v_x(x,y)\, i \\ &= e^x\cos y + i e^x \sin y \\ &= e^z \end{aligned}$$

Remark The complex exponential function has some properties that the real exponential function doesn't have:

① $f(z)=e^z$ has period $2\pi i$.

② $f(z)=e^z$ may be a negative real number.

For example. $f(\pi i)=e^{\pi i}=-1$

More generally, we have the following proposition.

Proposition (i) $\mathrm{Range}(e^z)=\mathbb{C}\setminus\{0\}$

(ii) If $e^{z_1}=e^{z_2}$, then $z_1-z_2=2\pi i\cdot k$ for some $k\in\mathbb{Z}$

Proof. (i) Given any complex number $z_0\neq 0$, we can write it in the form $z_0=re^{i\theta}$ for some $r>0$ and $\theta\in\mathbb{R}$.

Since $r>0$, there $\exists$ $x\in\mathbb{R}$ such that $r=e^x$.

$z_0=re^{i\theta}=e^x\cdot e^{i\theta}=e^{x+i\theta}=f(x+i\theta)$

Also, $e^z\neq 0$ for any $z\in\mathbb{C}$ since $|e^z|=|e^x|>0$.

(ii). Write $z_1=x_1+y_1i$, $z_2=x_2+y_2i$.

$e^{z_1}=e^{z_2}\Rightarrow e^{x_1}\cdot e^{y_1i}=e^{x_2}\cdot e^{y_2i}$

$|e^{z_1}|=|e^{z_2}|\Rightarrow e^{x_1}=e^{x_2}\Rightarrow x_1=x_2$.

So $e^{y_1i}=e^{y_2i}\Rightarrow e^{(y_1-y_2)i}=1\Rightarrow y_1-y_2=2\pi ik$ for some $k\in\mathbb{Z}$

THE LOGARITHMIC FUNCTION

Recall that in the previous section, we described that given $z \in \mathbb{C} \setminus \{0\}$, how to find $w \in \mathbb{C}$ such that $e^w = z$, and different solutions are differed by $2\pi i k$, $k \in \mathbb{Z}$.

More concretely, if we write $w = w_1 + w_2 i$, and $z = |z| e^{i\theta}$, then $e^w = z$ means $$e^{w_1} \cdot e^{w_2 i} = |z| \cdot e^{i\theta}.$$

So $$\begin{cases} e^{w_1} = |z| \\ e^{w_2 i} = e^{i\theta} \end{cases} \Rightarrow \begin{cases} w_1 = \ln|z| \\ w_2 = \theta + 2k\pi \in \arg(z) \end{cases}$$

We conclude w can be taken to be $w = \ln|z| + i \arg(z)$

Following the idea that logrithmic function should be the inverse of exponential function, we can define

$$f(z) = \log z = \ln|z| + i \arg(z) \quad (z \neq 0)$$

Note that this is not the "function" we usually refer to, and instead, it's a "multi-value function".

One way to make it into a single value function is to make a choice of $\arg(z)$ for each $z \in \mathbb{C} \setminus \{0\}$, and a natural choice is the principal argument $\operatorname{Arg}(z)$

Definition. If $z \neq 0$, the pincipal value of $\log z$ is $\operatorname{Log} z = \ln|z| + i \operatorname{Arg}(z)$

Remark. When restricted to positive real numbers, $\operatorname{Log} z$ agress with the real logarithmic function.

Example. $\log(-1) = \log|-1| + i \arg(-1) = (\pi + 2k\pi) i$. $k \in \mathbb{Z}$

$\operatorname{Log}(-1) = \pi i$.