Example. $f(z) = |z|^2 = (x^2+y^2) + 0i$

So $u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$.

The Cauchy-Riemann equations are not satisfied unless $(x,y) = (0,0)$, so $f(z)$ is not differentiable at $z \neq 0$.

When $(x,y) = (0,0)$ we see u and v have continuous partial derivatives in a neighbourhood of $(0,0)$, and the Cauchy-Riemann Equations are satisfied, so we can conclude $f'(z)$ exists at $z=0$

Example. $f(z) = x^3 + i(1-y)^3$

So $u(x,y) = x^3$, $v(x,y) = (1-y)^3$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Leftrightarrow \begin{cases} 3x^2 = -3(1-y)^2 \\ 0 = 0 \end{cases} \Leftrightarrow x^2 + (y-1)^2 = 0$$

$$\Leftrightarrow (x,y) = (0,1)$$

We see if $(x,y) \neq (0,1)$, the Cauchy-Riemann Equations are not satisfied. so $f'(z)$ doesn't exist if $z \neq i$.

If $(x,y) = (0,1)$. the Cauchy-Riemann Equations are satisfied, and u_x, u_y, v_x, v_y are continuous. so $f'(i)$ exists and $f'(i) = u_x(0,1) + v_x(0,1)i = 0$

ANALYTIC FUNCTIONS

Definition. A complex function f is analytic in an open set S if it has derivative everywhere in S. f is analytic at $z_0 \in \mathbb{C}$ if its analytic in some neighbourhood of z_0.

Definition. An entire function is a function that is analytic at each point of $\mathbb{C}$.

Example
- $f(z) = z^2$ is an entire function
- $f(z) = \frac{1}{z}$ is analytic at any nonzero point
- $f(z) = |z|^2$ is NOT analytic anywhere

The basic differentiation rules indicate the following:

Proposition ① If f and g are analytic functions on an open connected set D, then $f \pm g$, $f \cdot g$ are analytic on D, and $\frac{f}{g}$ is analytic in D provided g doesn't vanish on D

② If f is analytic on an open connected set D, and $f(D)$ is contained in an open connected set on which g is analytic, then $g \circ f$ is analytic on D.

Definition
- An open set $S \subseteq \mathbb{C}$ is connected if any pair of elements in S can be connected by a polygonal line in S, consisting of finitely number of segments.
- A non-empty open connected set is called a domain
- A domain with none, some, or all its boundary points is called a region

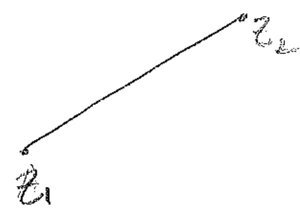
Proposition. If $f'(z)=0$ everywhere in a domain D, then f is a constant function on D

Proof

$f'(z) = U_x + V_x i = 0$ and $U_x = V_y$, $U_y = V_x$.

So $U_x = U_y = V_x = V_y = 0$ on D

We only need to show f is constant on any line segment in D, then since D is a domain, any two points are connected, we can therefore conclude f is constant throughout D.



We can parameterize a line segment by $(x(t), y(t))$,

so $\frac{d}{dt}U(x(t), y(t)) = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = 0$

it indicates $U(x(t), y(t))$ is a constant function.
So U is constant on the line segment.
Similarly, V is also constant

Corollary. If $f(z)$ and $\overline{f(z)}$ are both analytic inside a domain D, then $f(z) \equiv C \in \mathbb{C}$ on D.

Proof. If $f(z) = U(x,y) + V(x,y)i$, then $\overline{f(z)} = U(x,y) - V(x,y)i$

$f(z)$ is analytic on $D \Rightarrow \begin{cases} U_x = V_y \\ U_y = -V_x \end{cases}$ on D.

$\overline{f(z)}$ is analytic on $D \Rightarrow \begin{cases} U_x = (-V)_y = -V_y \\ U_y = -(-V)_x = V_x \end{cases}$ on D

So we get $u_x = u_y = v_x = v_y$ on D

which implies $f'(z) = u_x + v_x i = 0$ on D.

by the previous proposition, $f(z) \equiv C \in \mathbb{C}$ on D.

Corollary. If $f(z)$ is an analytic function on a domain D, with $|f(z)| \equiv r \in \mathbb{R}^{\geq 0}$ on D, then $f(z) \equiv C \in \mathbb{C}$ on D

Proof. If $r = 0$, then $|f(z)| \equiv 0 \Rightarrow f(z) \equiv 0$.

If $r > 0$. then $f(z) \cdot \overline{f(z)} = |f(z)|^2 = r^2$

so $\overline{f(z)} = \dfrac{r^2}{f(z)}$ is analytic, since

$f(z)$ is analytic.

By the previous Corollary, $f(z)$, $\overline{f(z)}$ both analytic on $D \Rightarrow f(z) \equiv C \in \mathbb{C}$ on D.

Definition. $H(x,y) : \mathbb{R}^2 \supseteq D \longrightarrow \mathbb{R}$ is called harmonic on D if

$$H_{xx} + H_{yy} = 0 \quad \text{throughout } D.$$

and H has continuous first and second order partial derivatives.

Theorem. If $f(z) = u(x,y) + v(x,y)i$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof. $f(z)$ is analytic $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{cases}$

So $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$.

Similarly we can prove $V_{xx} = V_{yy}$.

Remark. In the above proof, we do need to verify u and v have continuous first and second order derivatives, but the proof of this needs the theorem that

$f(z)$ is analytic $\Rightarrow$ $f(z)$ has derivatives of arbitrary order.

which shall be proved later in this course.

Example. $f(z) = \frac{1}{z^2}$ is analytic at any $z \neq 0$,

$$f(z) = \frac{1}{z^2} = \frac{1}{z^2} \cdot \frac{\bar{z}^2}{\bar{z}^2} = \frac{\bar{z}^2}{|z|^4} = \frac{(x^2-y^2)-2xyi}{(x^2+y^2)^2}$$

So $u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ and $v(x,y) = \frac{-2xy}{(x^2+y^2)^2}$ are harmonic functions on any domain not containing origin

Remark. Harmonic functions play an important role in analysis and applied math. For functions with n variables $u(x_1, \cdots, x_n)$. we define u to be harmonic if
$$u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0$$

Definition. If $u(x,y)$ is harmonic, we define $v(x,y)$ to be a harmonic conjugate of $u(x,y)$ if $f(z) = u(x,y) + v(x,y)i$ is analytic.

So in the above example, we see $V(x,y) = \frac{-2xy}{(x^2+y^2)^2}$ is a harmonic conjugate of $u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$

Exercise. Show that the harmonic conjugate of $u(x,y)$ on a domain D is unique up to adding a constant.