BASIC RULES FOR DIFFERENTIATION

Theorem If f and g are differentiable at $z \in \mathbb{C}$, then $f+g$, $f-g$, $f \cdot g$, $\frac{f}{g}$ (when $g(z) \neq 0$) are all differentiable at z, and :

(i) $(f+g)'(z) = f'(z) + g'(z)$

(ii) $(f-g)'(z) = f'(z) - g'(z)$

(iii) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

(iv) $\left(\frac{f}{g}\right)(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

Theorem. If $c \in \mathbb{C}$ is a constant, then

(i) $c' = 0$.

(ii) If f is differentiable at $z \in \mathbb{C}$, then cf is also differentiable at z with $(cf)'(z) = cf'(z)$

Theorem (Chain Rule) If f has derivative at $z_0 \in \mathbb{C}$, and g has a derivative at $f(z_0) \in \mathbb{C}$, then $F(z) = g(f(z))$ also has derivative at $z_0 \in \mathbb{C}$, with

$$F'(z_0) = g'(f(z_0))f'(z_0)$$

The proofs of the above theorems are almost identical to their counterparts in Calculus, so we omit the proofs here.

Example. By induction, we can verify that the power rule still holds for complex function: If n is a nonzero integer, then $(z^n)' = nz^{n-1}$

So for a polynomial $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$:

$$p'(z) = c_1 + 2c_2 z + 3c_3 z^2 + \cdots + n c_n z^{n-1}.$$

Example. $f(z) = \dfrac{z-1}{2z+1}$, then

$$f'(z) = \frac{(z-1)'(2z+1) - (z-1)(2z+1)'}{(2z+1)^2}$$

$$= \frac{(2z+1) - 2(z-1)}{(2z+1)^2}$$

$$= \frac{3}{(2z+1)^2}$$

CAUCHY-RIEMANN EQUATIONS

If $f(x+yi) = u(x,y) + v(x,y)i$ has derivative at $z_0 = x_0 + y_0 i$, we would like to find what conditions do $u(x,y)$, $v(x,y)$ need to satisfy.

If $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists,

We may choose to let $\Delta z \to 0$ along real-axis:

$$\lim_{\Delta x \to 0} \frac{f(x_0+y_0 i+\Delta x) - f(x_0+y_0 i)}{\Delta x}$$

$$= \left(\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right) i$$

$$= u_x(x_0, y_0) + v_x(x_0, y_0) i$$

We can also choose to let $\Delta z \to 0$ along imaginary-axis:

$$\lim_{\Delta y \to 0} \frac{f(x_0+y_0 i+\Delta y i) - f(x_0+y_0 i)}{\Delta y i}$$

$$= \left(\lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y\, i}\right) + \left(\lim_{\Delta y \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y\, i}\right) i$$

$$= u_y(x_0, y_0) \cdot \frac{1}{i} + v_y(x_0, y_0)$$

$$= v_y(x_0, y_0) - u_y(x_0, y_0) i$$

We therefore get $u_x(x_0, y_0) + v_x(x_0, y_0) i = v_y(x_0, y_0) - u_y(x_0, y_0) i$

which splits into a pair of equations:

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases}$$

The above equations are called the Cauchy-Riemann Equations, and we can make the conclusion:

Theorem. $f(z) = u(x,y) + v(x,y)i$, and $f'(z)$ exists at $z_0 = x_0 + y_0 i$.

Then the first order partial derivatives of u & v exist at (x_0, y_0), satisfying the Cauchy-Riemann Equations, $u_x = v_y$ and $u_y = -v_x$. What's more, we can write

$$f'(z_0) = u_x(x_0, y_0) + v_x(x_0, y_0)i$$

The above theorem tells us that the Cauchy-Riemann Equations are a necessary condition for f to have derivative at $z_0 \in \mathbb{C}$. But in general, this condition is NOT sufficient:

Example. $$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

So when $z \neq 0$, $f(z) = \frac{(\bar{z})^2}{z} = \frac{(x - yi)^2}{x + yi}$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + \frac{y^3 - 3x^2y}{x^2 + y^2} i$$

So $$u(x,y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$ and

$$v(x,y) = \begin{cases} \frac{y^3 - 3x^2y}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Then $u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

$$V_y(0,0) = \lim_{\Delta y \to 0} \frac{V(0,\Delta y) - V(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1$$

So $u_x(0,0) = V_y(0,0)$ Similarly we can also verify that

$u_y(0,0) = -V_x(0,0)$.

But, f doesn't have derivative at $(0,0)$. It can be proved by taking different paths towards $(0,0)$

In order to make the Cauchy-Riemann Equations to be sufficient conditions, we need to add more assumptions to the function:

Theorem. $f(z) = u(x,y) + v(x,y)i$ is defined throughout some neighbourhood of $z_0 = x_0 + y_0 i$, and suppose:

(i) u_x, u_y, V_x, V_y exist throughout the neighbourhood.

(ii) u_x, u_y, V_x, V_y are continuous at (x_0, y_0)

(iii) $\begin{cases} u_x(0,0) = V_y(0,0) \\ u_y(0,0) = -V_x(0,0) \end{cases}$

Then $f'(z_0)$ exists, and $f'(z_0) = u_x(x_0, y_0) + V_x(x_0, y_0) i$

Proof. Part (i) and (ii) together imply u and v are differentiable functions (Recall that for multivariable functions, differentiability and existence of partial derivatives are not equivalent)

Since u and v are differentiable in the neighbourhood,

$$\Delta u = u(x,y) - u(x_0,y_0) = u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\Delta v = v(x,y) - v(x_0,y_0) = v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y.$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

We then get

$$\begin{aligned} f(z_0+\Delta z) - f(z_0) &= \Delta u + \Delta v \cdot i \\ &= u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2 \Delta y \\ &\quad + [v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y]\, i \\ &= u_x(x_0,y_0)\Delta x - v_x(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\ &\quad + [v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y]\, i \\ &= u_x(x_0,y_0)\Delta z + v_x(x_0,y_0)\, i \cdot \Delta z \\ &\quad + (\varepsilon_1 + \varepsilon_3)\Delta x + (\varepsilon_2 + \varepsilon_4)\Delta y \end{aligned}$$

So
$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = u_x(x_0,y_0) + v_x(x_0,y_0)\, i + (\varepsilon_1 + \varepsilon_3)\cdot\frac{\Delta x}{\Delta z} + (\varepsilon_2 + \varepsilon_4)\cdot\frac{\Delta y}{\Delta z}$$

as $\Delta z \to 0$, we see

$$\lim_{z \to z_0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = u_x(x_0,y_0) + v_x(x_0,y_0)\, i$$

So $f'(z_0) = u_x(x_0,y_0) + v_x(x_0,y_0)\, i$