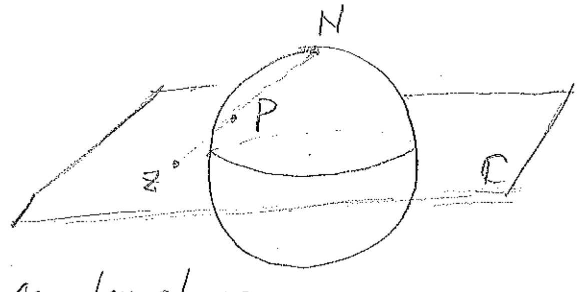
Consider A sphere, with a complex plane passing through its equator, and the point O coincides with the center of the sphere

Then for each point P on the sphere not the north pole N. The line passing through N and P intersects the complex plane at a point $z \in \mathbb{C}$. In this way, we can identify the points on the sphere not the north pole with complex numbers, and the north pole then corresponds to ∞.

This indicates the infinity ∞ can also be regarded as an element of complex numbers.



$\mathbb{C} \cup \{\infty\}$ is called the extened complex plane.

The Riemann Sphere motivates the concept of a "neighbourhood" of ∞. For each small $\varepsilon > 0$, the circle $r = \frac{1}{\varepsilon}$ on the complex plane corresponds to a small circle on the Riemann Sphere around the north pole. So $|z| > \frac{1}{\varepsilon}$ is regarded as a neighbourhood of ∞.

Definition. $\lim_{z \to \infty} f(z) = w$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|z| > \delta \Rightarrow |f(z) - w| < \varepsilon$$

Definition. $\lim_{z \to z_0} f(z) = \infty$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\varepsilon}$$

Theorem. If z_0 and w_0 are complex numbers, then

(i) $\lim_{z \to z_0} f(z) = \infty$ if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

(ii) $\lim_{z \to \infty} f(z) = w_0$ if $\lim_{z \to z_0} f(\frac{1}{z}) = w_0$

(iii) $\lim_{z \to \infty} f(z) = \infty$ if $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$

Example. $\lim_{z \to -1} \frac{z+1}{iz+3} = 0$, so $\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$.

$$\lim_{z \to \infty} \frac{2z+i}{z+1} = \lim_{z \to 0} \frac{\frac{2}{z}+i}{\frac{1}{z}+1} = \lim_{z \to 0} \frac{2+iz}{1+z} = 2$$

Definition. A function f is continuous at z_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

Proposition. f is continuous at z_0 if and only if:

(i) $\lim_{z \to z_0} f(z)$ exists

(ii) $f(z_0)$ is defined.

(iii) $f(z_0) = \lim_{z \to z_0} f(z)$

Theorem. Compositions of continuous functions results in a continuous function.

Theorem. If a function $f(z)$ is continuous and nonzero at z_0, then $f(z) \neq 0$ throughout some neighbourhood of that point.

The proofs of the above two theorems are similar to their counterpart in calculus / analysis, so we won't discuss about them here. You may try to prove it by recalling the proofs in calculus / analysis

Theorem. Write $f(z) = f(x+yi) = u(x,y) + v(x,y)i$; then

f is continuous at $x_0 + y_0 i$ iff u, v are continuous at (x_0, y_0)

Proof. It follows directly from the Proposition above.

Theorem. If f is continuous throughout a closed and bounded region R on $\mathbb{C}$, then $\max\limits_{z\in R} |f(z)|$ exists.

Proof. Write $f(x+yi) = u(x,y) + v(x,y)i$.

$$|f(x+yi)| = \sqrt{u(x,y)^2 + v(x,y)^2}$$

let $g(x,y) = \sqrt{u(x,y)^2 + v(x,y)^2}$, defined on

$R' = \{(x,y) \in \mathbb{R}^2 \mid x+yi \in R\}$. then by the corresponding result from multi-variable calculus, we know $\max\limits_{R'} g(x,y)$ exists, which is equivalent to $\max\limits_{R} |f(x+yi)|$ exists.

Example. Polynomials are continuous functions on all points of $\mathbb{C}$.

DERIVATIVES & DIFFERENTIATION

Definition. Let f be a complex function whose domain contains a neighbourhood $|z-z_0|<\varepsilon$ of $z_0 \in \mathbb{C}$. Define the derivative of f at z_0 to be

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

If the limit exists, we say f is differentiable at z_0.

Another way of expressing this limit is to write $\Delta z = z - z_0$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

So we can replace z_0 by z in the above expression:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

where $\Delta w = f(z+\Delta z) - f(z)$, if we consider $w = f(z)$.

Example. $f(z) = \frac{1}{z}$. At each nonzero z,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{-\Delta z}{(z+\Delta z)z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} -\frac{1}{(z+\Delta z)z}$$

$$= -\frac{1}{z^2}$$

so $f'(z) = \frac{1}{z}$.

Example. $f(z) = \overline{z}$.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

This limit doesn't exists (Recall we have proved a similar limit doesn't exist in previous section). So $f(z)$ is not differentiable at any point.

Example. $f(z) = |z|^2 = z \cdot \bar{z}$

$$\frac{\Delta w}{\Delta z} = \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\cdot\bar{z}}{\Delta z} = \frac{z\bar{z} + \Delta z\,\bar{z} + z\overline{\Delta z} + \Delta z\cdot\overline{\Delta z} - z\bar{z}}{\Delta z}$$

$$= \bar{z} + \overline{\Delta z} + z\cdot\frac{\overline{\Delta z}}{\Delta z}$$

If $\Delta z \to 0$ along positive real axis.

$$\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z \longrightarrow \bar{z} + z$$

If $\Delta z \to 0$ along positive imaginary axis.

$$\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z \longrightarrow \bar{z} - z$$

So if $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ exists, $\bar{z} + z = \bar{z} - z$, which implies $z = 0$

it indicates $f(z)$ is NOT differentiable at $z \neq 0$

when $z = 0$. $f'(0) = \lim\limits_{z\to 0} \frac{|z|^2 - |0|^2}{z} = \lim\limits_{z\to 0}\frac{z\cdot\bar{z}}{z} = \lim\limits_{z\to 0} \bar{z} = 0$

So $f(z)$ is only differentiable at $z = 0$.

Example. If f is a complex function such that $f(z) \in \mathbb{R} \subseteq \mathbb{C}$ for all $z \in \mathbb{C}$, and f is differentiable at $z_0 \in \mathbb{C}$, then $f'(z_0) = 0$.

Proof: $f'(z_0) = \lim\limits_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists.

if $\Delta z \to 0$ along real axis. $\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \in \mathbb{R}$

if $\Delta z \to 0$ along imaginary axis, $\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \in \mathbb{R}i$

So when taking the limit, we see
$f'(z_0) \in \mathbb{R} \cap \mathbb{R}i = \{0\}$

$f'(z_0) = 0$.