Corollary. If \( f \) is analytic throughout a neighbourhood \( N_0 \) of \( z_0 \), and there is a sequence \( \{z_n\} \) such that \( \lim_{n \to \infty} z_n = z_0 \), \( \forall n \in \mathbb{N}, z_n \neq z \) and \( f(z_n) = 0 \), then \( f(z) = 0 \) on \( N_0 \).

Proof. First, we see there exists a neighbourhood \( N \) of \( z_0 \) such that \( f(z) \equiv f(z_0) = f(\lim_{n \to \infty} z_n) = \lim_{n \to \infty} f(z_n) = 0 \), otherwise \( z_0 \) will be an isolated zero. Contradict to \( \lim_{n \to \infty} z_n = z_0 \) \& \( f(z_n) = 0 \). So the Taylor series expansion of \( f \) at \( z_0 \) is \( f(z) \equiv 0 \). And \( f(z) \) is also analytic on \( N \), the same Taylor series expansion \( f(z) \equiv 0 \) holds on \( N_0 \) as well.

Theorem (Coincidence Principle). A function \( f \) is analytic on a domain \( D \), and \( \{z_n\} \) is a sequence in \( D \) with \( \lim_{n \to \infty} z_n = z_0 \in D \) and \( z_n \neq z \) \( \forall n \in \mathbb{N} \). If \( f(z_n) = 0 \) \( \forall n \in \mathbb{N} \), then \( f(z) = 0 \) on \( D \).

Proof. For any \( z \in D \), connect \( z_0 \) \& \( z \) by a polygonal path \( L \). Let \( d \) be the shortest distance between \( \partial D \) and \( L \). Along \( L \) we pick points \( z_0 = s_0, s_1, s_2, \ldots \) such that \( |s_k - s_{k-1}| < d \), and consider the balls \( B_k = B(s_k, d) \). So each \( s_{k+1} \in B_k \). On \( B_0 \), by the previous corollary, we see \( f \equiv 0 \) on \( B_0 \), so it's zero on \( L \cap B_0 \), which tells us \( z \) is a limit of a sequence of zeros. So by the previous corollary again, we get \( f \equiv 0 \) on \( B_1 \). Continue this argument along \( L \), we can finish the proof.
Corollary. \( f \) and \( g \) are functions analytic on a domain \( D \). 
\( \{ z_n \} \) is a sequence in \( D \) such that \( \lim_{n \to \infty} z_n = z_0 \in D \), 
\( z_n \neq z_0 \forall n \in \mathbb{N} \). If \( f(z_n) = 0 \forall n \in \mathbb{N} \), then \( f(z) \equiv 0 \) on \( D \).

Proof. Take \( f(z) - g(z) \) as the function in the theorem.

Next we are discussing the relations between zeros & poles.

Theorem. If \( p(z) \) and \( q(z) \) are analytic at \( z_0 \), and \( p(z_0) \neq 0 \), \( q(z) \) has a zero of order \( m \) at \( z_0 \), then:
\[ \frac{p(z)}{q(z)} \] has a pole of order \( m \) at \( z_0 \).

Proof. \( q(z) \) has a zero of order \( m \) at \( z_0 \) implies
\[ q(z) = (z-z_0)^m g(z) \]
for some \( g(z) \) analytic at \( z_0 \) and \( g(z_0) \neq 0 \).

Then
\[ \frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)^m g(z)} = \frac{p(z)}{(z-z_0)^m} \]
Note that
\[ \frac{p(z)}{q(z)} \] is analytic at \( z_0 \) and \( \frac{p(z)}{q(z)} \neq 0 \) so
\[ \frac{p(z)}{q(z)} \] has a pole of order \( m \) at \( z_0 \).

Example. \( \frac{1}{1 - \cos z} \) has a pole of order \( 2 \) at \( z_0 = 0 \) since
we can let \( p(z) = 1 \), \( q(z) = 1 - \cos z \) in the above theorem.
\( p(z) = 1 \) is analytic & non-zero at \( z_0 = 0 \).
\( q(z) = 1 - \cos z \) has a zero of order \( 2 \) at \( z_0 = 0 \).
Theorem. \( p(z) \) and \( q(z) \) are analytic at \( z_0 \). If \( p(z_0) \neq 0 \), \( q(z_0) = 0 \) and \( q'(z_0) \neq 0 \), then \( z_0 \) is a simple pole of \( \frac{p(z)}{q(z)} \) and

\[
\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}
\]

Proof. \( q(z_0) = 0 \), \( q'(z_0) \neq 0 \) \( \Rightarrow \) \( q \) has a zero of order one at \( z_0 \). By the previous theorem, \( \frac{p(z)}{q(z)} \) has a simple pole at \( z_0 \).

Let \( q(z) = (z-z_0) g(z) \), \( g(z) \) is analytic at \( z_0 \), \( g(z_0) \neq 0 \).

Then \( \frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0) g(z)} = \frac{p(z)}{g(z)} \left( \frac{1}{z-z_0} \right) \)

\[
\text{Res}_{z=z_0} \left( \frac{p(z)}{g(z)} \right) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)} \quad \text{(Since \( q'(z) = g(z) + (z-z_0) g'(z) \))}
\]

Example. \( f(z) = \frac{z}{z^4 + 4} \). \( z_0 = \sqrt[4]{2} e^{\frac{\pi}{4} i} \) is a pole of \( f \).

Let \( p(z) = z \), \( q(z) = z^4 + 4 \), we see \( p(z_0) \neq 0 \), \( q(z_0) = 0 \)

but \( q'(z_0) = 4z_0^3 \neq 0 \), so by the theorem we get

\[
\text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4} \cdot \frac{1}{z_0^2} = \frac{1}{8} \cdot e^{-\frac{\pi}{2} i} = -\frac{i}{8}
\]
Theorem. If \( z_0 \) is a removable singular point of \( f \), then \( f \) is bounded and analytic on some deleted neighbourhood \( 0 < |z - z_0| < \epsilon \).

Proof. There exists \( F(z) \) such that
\[ F(z) = f(z) \quad \text{for} \quad z \neq z_0 \quad \text{and} \quad F(z) \text{ is analytic at } z_0. \]

Since \( z_0 \) is an isolated singular point, we can find some \( \epsilon > 0 \) such that \( F(z) = f(z) \) is analytic on \( 0 < |z - z_0| < \epsilon \), so \( F(z) \) is analytic on \( 0 < |z - z_0| < \epsilon \), hence continuous.

\( F(z) \) is therefore bounded on \( 1 < |z - z_0| < \frac{\epsilon}{2} \), a closed disk, so also bounded on the subset \( 0 < |z - z_0| < \frac{\epsilon}{2} \), which implies \( f(z) \) is also bounded on \( 0 < |z - z_0| < \frac{\epsilon}{2} \).

Theorem (Riemann's Theorem).
Suppose \( f \) is bounded and analytic on \( 0 < |z - z_0| < \epsilon \). If \( f \) is not analytic at \( z_0 \), then \( z_0 \) is a removable singular point.

Proof. \( f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad (0 < |z - z_0| < \epsilon) \)

Let \( 0 < r < \epsilon \), and \( C \) be the positively oriented contour \( |z - z_0| = r \).

Assume \( f \) is bounded by \( M \) on \( 0 < |z - z_0| < \epsilon \),
We know \( b_n = \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \) 

\[ |b_n| \leq \frac{1}{2\pi} \cdot M \cdot \frac{r^{-n+1}}{r^{n+1}} = M \cdot r^n \]

Since the above holds for any \( 0 < r < \varepsilon \). Letting \( r \to 0 \), we get \( |b_n| = 0 \Rightarrow b_n = 0 \)

**Theorem. (Casorati-Weierstrass Theorem)**

\( z_0 \) is an essential singular point of \( f \). For any \( \omega_0 \in \mathbb{C} \), any \( \varepsilon > 0 \) and any \( \delta > 0 \), there exists \( 0 < |z - z_0| < \delta \) such that \( |f(z) - \omega_0| < \varepsilon \).

**Proof.** Suppose we can find \( \omega_0 \in \mathbb{C} \), \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( \forall 0 < |z - z_0| < \delta \), \( |f(z) - \omega_0| \geq \varepsilon \).

By taking even small \( \delta \) if necessary, we can assume \( f \) is analytic on \( 0 < |z - z_0| < \delta \).

Define \( G(z) = \frac{1}{f(z) - \omega_0} \) \( (0 < |z - z_0| < \delta) \)

\[ |G(z)| = \frac{1}{|f(z) - \omega_0|} < \frac{1}{\varepsilon} \text{ on } 0 < |z - z_0| < \delta, \text{ so} \]

\( g(z) \) is bounded and analytic on \( 0 < |z - z_0| < \delta \), applying Riemann's Theorem, \( z_0 \) is a removable singular point of \( f \).

So we can extend \( g(z) \) to \( \widehat{G}(z) \), which is analytic on \( |z - z_0| < \delta \).
If \( G(z) \neq 0 \), define \( F(z) = \frac{1}{G(z)} + W_0 \) on \( |z - z_0| < \delta \).

When \( 0 < |z - z_0| < \delta \), observe that

\[
F(z) = \frac{1}{g(z)} + W_0 \Rightarrow g(z) = \frac{1}{F(z) - W_0},
\]

so \( F(z) = f(z) \) on \( 0 < |z - z_0| < \delta \), and \( F(z) \) is analytic at \( z_0 \).

We see \( z_0 \) is a removable singular point of \( f(z) \), contradiction.

If \( G(z_0) = 0 \), since \( G(z) \) is not constant, \( z_0 \) is a zero of some finite order \( m \) of \( G \).

\[
s(z) = \frac{1}{g(z)} + W_0 = \frac{1 + W_0 G(z)}{G(z)} \]

\( 1 + W_0 G(z) = 1 \neq 0 \), and \( G(z) \) has zero of order \( m \) at \( z_0 \).

So \( f(z) \) has a pole of order \( m \) at \( z_0 \), contradiction.

**Theorem:** If \( z_0 \) is a pole of \( f \), then \( \lim_{z \to z_0} f(z) = \infty \).

**Proof:** Assume \( z_0 \) is a pole of order \( m \), then

\[
f(z) = \frac{\phi(z)}{(z - z_0)^m}
\]

for some \( \phi(z) \) analytic at \( z_0 \) and \( \phi(z_0) \neq 0 \).

\[
\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{0}{\phi(z_0)} = 0
\]

Since \( \phi(z) \) is continuous at \( z_0 \).

We get \( \lim_{z \to z_0} f(z) = \infty \).