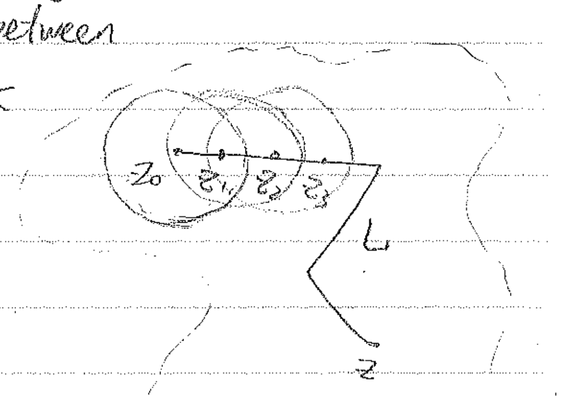
Corollary. If f is analytic throughout a neighbourhood N_0 of z_0, and there is a sequence $\{z_n\}$ such that $\lim_{n\to\infty} z_n = z_0$, $\forall n \in \mathbb{N}$, $z_n \neq z$ and $f(z_n)=0$, then $f(z) \equiv 0$ on N_0.

Proof. First, we see there exits neighbourhood N of z_0 such that $f(z) \equiv f(z_0) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = 0$, otherwise z_0 will be an isolated zero, contradict to $\lim_{n\to\infty} z_n = z_0$ & $f(z_n)=0$

So the Taylor series expansion of f at z_0 is $f(z)\equiv 0$.
And $f(z)$ is also analytic on $N_0 \ni z_0$, the same taylor series expansion $f(z)\equiv 0$ holds on N_0 as well.

Theorem (Coincidence Principle) A function f is analytic on a domain D, and $\{z_n\}$ is a sequence in D with $\lim_{n\to\infty} z_n = z_0 \in D$ and $z_n \neq z_0$ $\forall n \in \mathbb{N}$. If $f(z_n)=0$ $\forall n \in \mathbb{N}$, then $f(z)\equiv 0$ on D

Proof For any $z \in D$, connect z_0 & z by a polygonal path L.
Let d be the shortest distance between ∂D and L. Along L we pick points $z_0 = s_0, s_1, s_2, \ldots, s_n$ such that $|s_k - s_{k-1}| < d$, and consider the balls $B_k = B(s_k, d)$ so each $s_{k+1} \in B_k$



On B_0, by the previous corollary, we see $f \equiv 0$ on B_0, so it's zero on $L \cap B_0$, which tells us z_1 is a limit of a sequence of zeros, so by the previous corollary again, we get $f \equiv 0$ on B_1. Continue this argument along L, we can finish the proof

Corollary. f and g are functions analytic on a domain D. $\{z_n\}$ is a sequence in D such that $\lim_{n\to\infty} z_n = z_0 \in D$, $z_n \neq z_0 \ \forall n \in \mathbb{N}$. If $f(z_n) = 0 \ \forall n \in \mathbb{N}$, then $f(z) \equiv 0$ on D

Proof. Take $f(z) - g(z)$ as the function in the Theorem

Next we are discussing the relations between zeros & poles

Theorem. If $p(z)$ and $q(z)$ are analytic at z_0, and $p(z_0) \neq 0$. $q(z)$ has a zero of order m at z_0, then:

$\frac{p(z)}{q(z)}$ has a pole of order m at z_0

Proof $q(z)$ has a zero of order m at z_0 implies

$$q(z) = (z-z_0)^m g(z)$$

for some $g(z)$ analytic at z_0 and $g(z_0) \neq 0$

Then $\frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)^m g(z)} = \frac{\frac{p(z)}{g(z)}}{(z-z_0)^m}$, Note that

$\frac{p(z)}{g(z)}$ is analytic at z_0 and $\frac{p(z_0)}{g(z_0)} \neq 0$. so

$\frac{p(z)}{q(z)}$ has a pole of order m at z_0.

Example. $\frac{1}{1-\cos z}$ has a pole of order 2 at $z_0 = 0$ since

we can let $p(z) \equiv 1$, $q(z) = 1 - \cos z$ in the above theorem:

$p(z) \equiv 1$ is analytic & nonzero at $z_0 = 0$.

$q(z) = 1 - \cos z$ has a zero of order 2 at $z_0 = 0$

Theorem. $p(z)$ and $q(z)$ are analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$, then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof. $q(z_0) = 0$, $q'(z_0) \neq 0 \Rightarrow q$ has a zero of order one at z_0.
by the previous theorem, $\frac{p(z)}{q(z)}$ has a simple pole at z_0

Let $q(z) = (z-z_0)g(z)$, $g(z)$ is analytic at z_0, $g(z_0) \neq 0$.
Then $\frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)g(z)} = \frac{\frac{p(z)}{g(z)}}{z-z_0}$

$$\operatorname*{Res}_{z=z_0} \left(\frac{p(z)}{q(z)}\right) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)} \quad \left(\text{since } q'(z) = g(z) + (z-z_0)g'(z)\right)$$

Example. $f(z) = \frac{z}{z^4+4}$. $z_0 = \sqrt{2}e^{\frac{\pi}{4}i}$ is a pole of f.

let $p(z) = z$, $q(z) = z^4+4$, we see $p(z_0) \neq 0$, $q(z_0) = 0$
but $q'(z_0) = 4z_0^3 \neq 0$, so by the theorem we get

$$\operatorname*{Res}_{z \to z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8} \cdot e^{-\frac{\pi}{2}i} = -\frac{i}{8}$$

BEHAVIOUR NEAR SINGULARITIES

Theorem. If z_0 is a removable singular point of z_0, then f is bounded and analytic on some deleted neighbourhood $0<|z-z_0|<\varepsilon$.

Proof. There exists $F(z)$ such that $F(z)=f(z)$ for $z\neq z_0$ and $F(z)$ is analytic at z_0.

Since z_0 is an isolated singular point we can find some $\varepsilon>0$ such that $F(z)=F(z_0)$ is analytic on $0<|z-z_0|<\varepsilon$ so $F(z)$ is analytic on $0<|z-z_0|<\varepsilon$, hence continuous, $F(z)$ is therefore bounded on $|z-z_0|\leq\frac{\varepsilon}{2}$, a closed disk so also bounded on the subset $0<|z-z_0|<\frac{\varepsilon}{2}$, which implies $f(z)$ is also bounded on $0<|z-z_0|<\frac{\varepsilon}{2}$.

Theorem (Riemann's Theorem)
Suppose f is bounded and analytic on $0<|z-z_0|<\varepsilon$. If f is not analytic at z_0, then z_0 is a removable singular point.

Proof. $$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{b=1}^{\infty}\frac{b_n}{(z-z_0)^n} \quad (0<|z-z_0|<\varepsilon)$$

Let $0<r<\varepsilon$, and C be the positively oriented contour $|z-z_0|=r$.

Assume f is bounded by M on $0<|z-z_0|<\varepsilon$.

We know $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz$

$|b_n| \le \frac{1}{2\pi}\cdot M\cdot\frac{1}{r^{-n+1}}\cdot 2\pi r = M\cdot r^n$

Since the above holds for any $0<r<\varepsilon$, letting $r\to 0$, we get $|b_n|=0 \Rightarrow b_n=0$

Theorem. (Casorati-Weierstrass Theorem)
z_0 is an essential singular point of f. For any $w_0\in\mathbb{C}$, any $\varepsilon>0$ and any $\delta>0$, there exists $0<|z-z_0|<\delta$ such that $|f(z)-w_0|<\varepsilon$.

Proof. Suppose we can find $w_0\in\mathbb{C}$, $\varepsilon>0$ and $\delta>0$ such that $\forall\ 0<|z-z_0|<\delta$, $|f(z)-w_0|\ge\varepsilon$.

By taking even small δ if necessary, we can assume f is analytic on $0<|z-z_0|<\delta$.

Define $g(z)=\frac{1}{f(z)-w_0}\quad(0<|z-z_0|<\delta)$

$|g(z)|=\frac{1}{|f(z)-w_0|}\le\frac{1}{\varepsilon}$ on $0<|z-z_0|<\delta$, so

$g(z)$ is bounded and analytic on $0<|z-z_0|<\delta$, applying Riemann's Theorem, z_0 is a removable singular point of g. So we can extend $g(z)$ to $G(z)$, which is analytic on $|z-z_0|<\delta$.

If $G(z_0) \neq 0$, define $F(z) = \frac{1}{G(z)} + w_0$ on $|z - z_0| < \delta$.

When $0 < |z - z_0| < \delta$, observe that

$$F(z) = \frac{1}{g(z)} + w_0 \Rightarrow g(z) = \frac{1}{F(z) - w_0},$$

So $F(z) = f(z)$ on $0 < |z - z_0| < \delta$, and $F(z)$ is analytic at z_0, we see z_0 is a removable singular point of $f(z)$, contradiction.

If $G(z_0) = 0$, since $G(z)$ is not constant, z_0 is a zero of some finite order m of G.

$$\therefore f(z) = \frac{1}{g(z)} + w_0 = \frac{1 + w_0 G(z)}{G(z)}$$

$1 + w_0 G(z_0) = 1 \neq 0$, and $G(z)$ has zero of order m at z_0, so $f(z)$ has a pole of order m at z_0, contradiction.

Theorem. If z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$

Proof. Assume z_0 is a pole of order m. then

$f(z) = \frac{\phi(z)}{(z - z_0)^m}$ for some $\phi(z)$ analytic at z_0 and $\phi(z_0) \neq 0$

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{0}{\phi(z_0)} = 0$$

since $\phi(z)$ is continuous at z_0.

We get $\lim_{z \to z_0} f(z) = \infty$