Theorem. z_0 is an isolated singular point of f. The following are equivalent:

(i) z_0 is a pole of order m of f

(ii) If f is analytic on $0<|z-z_0|<R$, then there exists a function ϕ analytic on $|z-z_0|<R$ such that $\phi(z_0) \neq 0$ and $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$

Proof. (i) $\Rightarrow$ (ii): If z_0 is a pole of order m, and f is analytic on $0<|z-z_0|<R$, the Laurant series on $0<|z-z_0|<R$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m} \quad (0<|z-z_0|<R)$$

with $b_m \neq 0$.

Define $\phi(z) = \begin{cases} b_m & \text{if } z = z_0 \\ (z-z_0)^m f(z) & \text{if } 0<|z-z_0|<R. \end{cases}$

We see $\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \cdots + b_{m-1}(z-z_0) + b_m$

for $0<|z-z_0|<R$, and this equality also holds at z_0.

So $\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \cdots + b_{m-1}(z-z_0) + b_m$

is a power series convergent throughout $|z|<R$, we get $\phi(z)$ is analytic on $|z|<R$ with $\phi(z_0) = b_m \neq 0$,

and $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ on $0<|z-z_0|<R$

(ii) $\Rightarrow$ (i): If $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ on $0<|z-z_0|<R$.

- since $\phi(z)$ is analytic on $|z-z_0|<R$,

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n = \phi(z_0) + \sum_{n=1}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \text{ on } |z-z_0|<R$$

so $$f(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

$$= \frac{\phi(z_0)}{(z-z_0)^m} + \sum_{n=1}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \quad (0<|z-z_0|<R)$$

Recall that the assumption is $\phi(z_0) \neq 0$, we see f has a pole of order m at z_0.

Corollary. If $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ on $0<|z-z_0|<R$ for some ϕ analytic on $|z-z_0|<R$ with $\phi(z_0) \neq 0$, then

$$\operatorname*{Res}_{z=z_0}(f) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Proof. By the proof of the Theorem, we see

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \sum_{n=1}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \text{ on } 0<|z-z_0|<R$$

So $$\operatorname*{Res}_{z=z_0}(f) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Example. When the pole at z_0 is a simple pole, $m=1$ so the above formula reduces to

$$\operatorname*{Res}_{z=z_0}(f) = \phi(z_0)$$

Example. $f(z) = \dfrac{z+4}{z^2+1}$ has an isolated singularity at $z_0 = i$

$$f(z) = \frac{\phi(z)}{z-i}, \text{ where } \phi(z) = \frac{z+4}{z+i} \text{ is analytic}$$

at $z_0 = i$, and $\phi(i) \neq 0$, so

$$\operatorname*{Res}_{z=i}(f) = \phi(i) = \frac{4+i}{2i}$$

Example. $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ has a pole at $z_0 = i$

$$f(z) = \frac{\phi(z)}{(z-i)^3}, \quad \phi(z) = z^3+2z \text{ is analytic at } z_0 = i,$$

and $\phi(i) \neq 0$. So the pole $z_0 = i$ is of order 3

$$\operatorname*{Res}_{z=i}(f) = \frac{\phi^{(3-1)}(i)}{(3-1)!} = \frac{\phi''(i)}{2!} = 3i$$

Example. $f(z) = \dfrac{(\log z)^3}{z^2+1}$, $\log z$ is the branch $0 < \theta < 2\pi$.

$$f(z) = \frac{\phi(z)}{z-i}, \text{ where } \phi(z) = \frac{(\log z)^3}{z+i}.$$

$\phi(z)$ is analytic at $z_0 = i$ and $\phi(i) = \dfrac{(\frac{\pi}{2}i)^3}{2i} = -\dfrac{\pi^3}{16} \neq 0$.

so $f(z)$ has a simple pole at $z_0 = i$. and

$$\operatorname*{Res}_{z=i}(f) = \phi(i) = -\frac{\pi^3}{16}$$

ZEROS OF ANALYTIC FUNCTIONS

Definition. If f is analytic at z_0, and there exists $m \in \mathbb{N}$ such that $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$, $f^{(m)}(z_0) \neq 0$, then we say z_0 is a zero of order m of f.

Example. $f(z) = (z-1)^2$ has a zero of order 2 at $z_0 = 1$

Theorem. f is analytic at z_0, then the following are equivalent:

(i) f has zero of order m at z_0

(ii) $\exists\, g(z)$, analytic and nozero at z_0 such that

$$f(z) = (z-z_0)^m g(z).$$

Proof. (i) $\Rightarrow$ (ii). If z_0 is a zero of order m, the Taylor series expansion of f at z_0 is:

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n,$$

where $|z-z_0| < \varepsilon$ for some $\varepsilon > 0$. such that f is analytic on $|z-z_0| < \varepsilon$

Let $g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n$, since it converges on $|z-z_0| < \varepsilon$, so it's analytic on $|z-z_0| < \varepsilon$, and

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

(ii) $\Rightarrow$ (i). If $f(z) = (z-z_0)^m g(z)$ for some $g(z)$ analytic at z_0 and $g(z_0) \neq 0$, then:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{on} \quad |z-z_0|<\varepsilon, \quad g(z_0)\neq 0.$$

$$\text{so} \quad f(z) = (z-z_0)^m g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n+m}$$

which is the Taylor expansion for f at z_0, we see

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \; ; \; f^{(m)}(z_0) \neq 0 .$$

so z_0 is a zero of order m.

Example. $f(z) = z^3 - 1$, $f(1) = 0$, $f'(1) = 3 \neq 0$, so $z_0 = 1$ is a zero of order 1.

$$f(z) = (z-1)g(z) \quad \text{where} \quad g(z) = z^2+z+1$$

Proposition. If f is analytic at z_0, and f is not constant on any neighbourhood of z_0, then there is a neighbourhood $0<|z-z_0|<\varepsilon$ such that $f(z) \neq f(z_0)$ on this neighbourhood

Proof. Let $F(z) = f(z) - f(z_0)$. then $F(z_0) = 0$ and F is analytic at z_0, F not constant on any neighbourhood of z_0.,
so we see not all $F^{(n)}(z_0) \neq 0$, otherwise the Taylor expansion of F at z_0 indicates $F(z) \equiv 0$ on a neighbourhood of z_0

Let the order of z_0 be m, then $\exists\, g$ such that $g(z)$ is analytic at z_0, $g(z_0) \neq 0$, and

$$F(z) = (z-z_0)^m g(z)$$

Note that $g(z_0) \neq 0$, so by continuity, $g(z) \neq 0$ on $|z-z_0|<\varepsilon$ for some $\varepsilon > 0$., so $F(z) \neq 0$ on $0<|z-z_0|<\varepsilon$, i.e. $f(z) \neq f(z_0)$