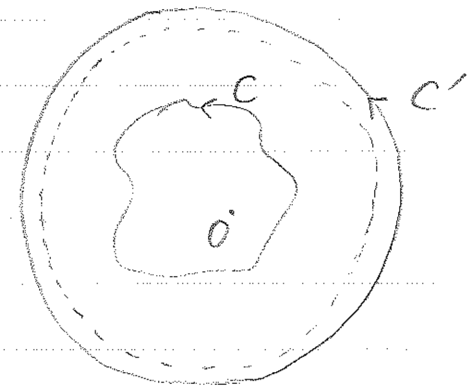
Theorem. If f is analytic everywhere in $\mathbb{C}$ except for a finite number of singular points interior to a positively oriented simple closed contour C, then:

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Proof. Take a positively oriented circle C': $|z|=R$ that encloses C in its interior.



then $f(z)$ is analytic on $|z|>R-\varepsilon$ which implies $f(\frac{1}{z})$ is analytic on $0<z<\frac{1}{R}$.

so 0 is an isolated singular point for $f(\frac{1}{z})$

Assume the Laurant series for $f(z)$ on $|z|>R$ is

$f(z) = \sum_{-\infty}^{+\infty} C_n z^n$, where $C_n = \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z^{n+1}}\,dz$.

then for $0<z<\frac{1}{R}$,

$$f\left(\frac{1}{z}\right) = \sum_{-\infty}^{+\infty} C_n z^{-n}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{-\infty}^{+\infty} C_n z^{-n-2}$$

We see $\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f(\frac{1}{z})\right] = C_{-1}$

So $\int_C f(z)\,dz = \int_{C'} f(z)\,dz = 2\pi i\, C_{-1} = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f(\frac{1}{z})\right]$

Remark. We define $\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$

Example. $f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$, C is the positively oriented circle $|z|=2$

If z_0 is a singular point of $f(z)$, then either $z_0=-1$ or $1+2z_0^4=0$

the latter implies $|z_0|^4 = |-\frac{1}{2}| = \frac{1}{2}$, so $|z_0|<1$.

We thus see all the singular points are inside C

so by the Theorem

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\cdot\frac{\left(\frac{1}{z}\right)^3\cdot\left(1-3\cdot\frac{1}{z}\right)}{\left(1+\frac{1}{z}\right)\left(1+2\cdot\left(\frac{1}{z}\right)^4\right)}$$

$$= \frac{1}{z^2}\cdot\frac{\frac{1}{z^3}\cdot\frac{z-3}{z}}{\frac{z+1}{z}\cdot\frac{2+z^4}{z^4}}$$

$$= \frac{1}{z}\cdot\frac{z-3}{(z+1)(2+z^4)}$$

Note $\frac{z-3}{(z+1)(2+z^4)}$ is analytic at $z_0=0$, so

$$\frac{1}{z}\cdot\frac{z-3}{(z+1)(2+z^4)} = \frac{1}{z}\cdot\left(\frac{0-3}{(0+1)(2+0^4)} + \text{positive power terms}\right)$$

So $\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2}$

we conclude $\int_C f(z)\,dz = 2\pi i\cdot\left(-\frac{3}{2}\right) = -3\pi i$

CLASSIFICATION OF ISOLATED SINGULARITIES

If z_0 is an isolated singular point for a function f, we know that in a deleted neighbourhood $0<|z-z_0|<\varepsilon$, we can express $f(z)$ in terms of Laurant Series:

$$f(z)=\sum_{n=-\infty}^{+\infty} C_n(z-z_0)^n$$

Definition. Under the above assumption, we say z_0 is a:
(i) Removable Singular Point if $C_n=0\ \forall\ n<0$
(ii) Pole if $C_n\neq 0$ for finitely many but at least one $n<0$
(iii) Essential Singular Point if $C_n\neq 0$ for infinitely many $n<0$

Now we will discuss each of the three cases.

Proposition. If z_0 is a removable singularity, then the function

$$g(z)=\begin{cases} C_0, & z=z_0 \\ f(z), & z\neq z_0 \end{cases}$$

is analytic at z_0.

Proof. Since $g(z)=f(z)=\sum_{n=-\infty}^{+\infty} C_n(z-z_0)^n=\sum_{n=0}^{\infty} C_n(z-z_0)^n$ for $0<|z-z_0|<\varepsilon$ for z_0 a removable singularity, and by our construction, $g(z_0)=C_0$ also agrees with the power series, we get

$$g(z)=\sum_{n=0}^{\infty} C_n(z-z_0)^n \text{ for all } |z-z_0|<\varepsilon.$$

So $g(z)$ is analytic on $|z-z_0|<\varepsilon$, hence analytic at z_0

Remark. In other words, z_0 is a removable singularity means if we adjust the value $f(z_0)$, we can make z_0 a regular point (i.e not a singularity)

Example. $f(z) = \begin{cases} 1, & z=0 \\ z, & z \neq 0 \end{cases}$ has a removable singularity at $z_0 = 0$.

~~If~~ we adjust $f(0)$ to be 0, then $z_0 = 0$

Exercise. $f(z) = \frac{\sin z}{z}$ has a removable singular point $z_0 = 0$

Definition. If z_0 is a pole for f and $C_{-m} \neq 0$, $C_{-n} = 0 \ \forall n > m$, we say z_0 is a pole of order m

By definition, if z_0 is a pole of order m, then the Laurant expansion near z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} b_n \frac{1}{(z-z_0)^n} \qquad (0 < |z-z_0| < \varepsilon)$$

Definition. A pole of order 1 is called a simple pole.

Example. $f(z) = \frac{1}{z}$ has a simple pole $z_0 = 0$

$f(z) = \frac{1}{z^2} + \frac{1}{z} + z^3$ has a pole of order 2 at $z_0 = 0$

Example. $f(z) = e^{\frac{1}{z}}$ has an essential singular point at $z_0 = 0$ since the Laurant expansion around 0 is

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n} \qquad (0 < |z| < \infty)$$