RESIDUES

Definition. If $f(z)$ is not analytic at z_0, but analytic at some point in every neighbourhood of it, we call z_0 a singular point of f

Definition. A singular point z_0 of f is isolated if there's a deleted ε-neighbourhood $0<|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic

Example. $z_0=0$ is a singular point for any branch of $f(z)=\log z$, but z_0 is not an isolated singular point since any neighbourhood $0<|z-z_0|<\varepsilon$ contains branch cut points

Example. $f(z)=\frac{1}{z}$ has an isolated singular point at $z_0=0$

Example. $f(z)=\dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$

The singular points are $z_0=0$ or $\frac{\pi}{z}=k\pi,\ k\in\mathbb{Z}$

i.e. $z_0=0$ or $z=\frac{1}{k},\ k\in\mathbb{Z}$

So we get $z_0=\frac{1}{k},k\in\mathbb{Z}$ are isolated singular points.

$z_0=0$ is a singular point that is not isolated

Lemma. If there're only finitely many singular points for $f(z)$ on a domain $\mathcal{U}$, then these singular points are all isolated.

Proof. Let $z_1,\dots,z_n$ be the set of singular points.

let $L_k=\min|z_k-z_i|$, then $f(z)$ is analytic on $0<|z-z_k|<L_k$, so z_k is isolated singular point.

Remark. ∞ is considered as an isolated singular point if $\exists\, R>0$ such that f is analytic on $|z|>R$.

Recall: If f is analytic on $U: 0<|z-z_0|<R$ for some $R>0$ and $f(z)=\sum_{n=-\infty}^{+\infty} C_n (z-z_0)^n$ on U, C is a positively oriented simple closed contour with z_0 in its interior, then

$$\int_C f(z)\, dz = 2\pi i\, C_{-1}$$

Definition. When z_0 is an isolated singularity of f, and the Laurant series expansion for f on $0<|z-z_0|<R$ for some $R>0$ is $f(z)=\sum_{n=-\infty}^{+\infty} C_n(z-z_0)^n$, then define the residue of f at z_0 to be $\underset{z=z_0}{\mathrm{Res}}(f)=C_{-1}$

Example. Consider $\int_C \frac{e^z-1}{z^4}\,dz$ where C is the positively oriented circle $|z|=1$:

$$\frac{e^z-1}{z^4}=\frac{1}{z^4}(e^z-1)=\frac{1}{z^4}\left(\sum_{n=1}^{\infty}\frac{1}{n!}z^n\right)=\sum_{n=1}^{\infty}\frac{1}{n!}z^{n-4}$$

$$=\frac{1}{z^3}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z}+\cdots$$

so $\underset{z=0}{\mathrm{Res}}(f)=\frac{1}{3!}=\frac{1}{6}$

We get $\int_C \frac{e^z-1}{z^4}\,dz=2\pi i\cdot\frac{1}{6}=\frac{\pi i}{3}$

Example. $\int_C \frac{dz}{z(z-2)^5}$ where C is the positively oriented circle $|z-2|=1$.

Note this circle is inside $0<|z-2|<2$, on which $f(z)=\frac{1}{z(z-2)^5}$ is analytic. so the Laurant expansion on $0<|z-2|<2$ is:

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5}\cdot\frac{1}{z} = \frac{1}{(z-2)^5}\cdot\frac{1}{2+(z-2)}$$

$$= \frac{1}{(z-2)^5}\cdot\frac{1}{2}\cdot\frac{1}{1+\frac{z-2}{2}}$$

$$= \frac{1}{2(z-2)^5}\cdot\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\cdot(z-2)^{n-5}$$

we see $C_{-1}=\frac{(-1)^4}{2^{4+1}}=\frac{1}{32}$

so $\int_C f(z)\,dz = 2\pi i\cdot\frac{1}{32} = \frac{\pi i}{16}$

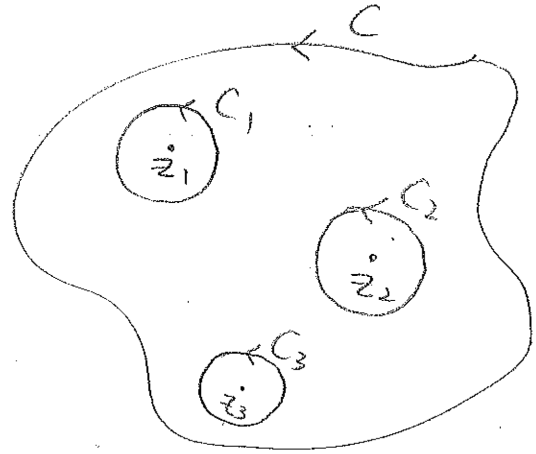
The above method of computing integrals by residue can be generalized by applying the multiply-connected version of Cauchy-Goursat Theorem, we get Cauchy's Residue Theorem:

Theorem. C is a simple closed contour positively oriented. If $f(z)$ is analytic on and inside C except for finitely many singular points $z_1,\dots,z_n$ inside C, then

$$\int_C f(z)\,dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k}(f)$$

Proof. Take positively oriented circles C_k around z_k such that they're small enough to be inside C, and they bound n disjoint disks.

Then by the multiply connected version of Cauchy-Goursat Theorem,



$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

$$= \sum_{k=1}^{n} 2\pi i \cdot \operatorname*{Res}_{z=z_k}(f)$$

$$= 2\pi i \sum_{k=0}^{n} \operatorname*{Res}_{z=z_k}(f)$$

Example. Let's compute $\int_C \frac{4z-5}{z(z-1)}\,dz$, where C is given by the circle $|z|=2$, positively oriented.

Observe that inside C there're two singular points: 0 and 1.

$$\frac{4z-5}{z(z-1)} = \left(4-\frac{5}{z}\right)\cdot\frac{1}{z-1} = \left(4-\frac{5}{z}\right)(-1-z-z^2-\cdots) \qquad (0<|z|<1)$$

$$\text{so } \operatorname*{Res}_{z=0}(f) = (-5)\times(-1) = 5.$$

$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z-1}\cdot\frac{1}{z} = \left(4-\frac{1}{z-1}\right)\cdot\frac{1}{1-(1-z)} = \left(4-\frac{1}{z-1}\right)\cdot\left(1-(z-1)+\cdots\right)$$

$$(0<|z-1|<1)$$

$$\text{so } \operatorname*{Res}_{z=1}(f) = -1$$

We get $\int_C f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=0}(f) + \operatorname*{Res}_{z=1}(f)\right) = 2\pi i\,(5-1) = 8\pi i$