LAURANT SERIES

Theorem. (Laurant's Theorem) f is analytic on an annular domain $R_1 < |z-z_0| < R_2$. C is any positively oriented simple closed contour around z_0 in that domain. Then for each z in $R_1 < |z-z_0| < R_2$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

Another way of writing is

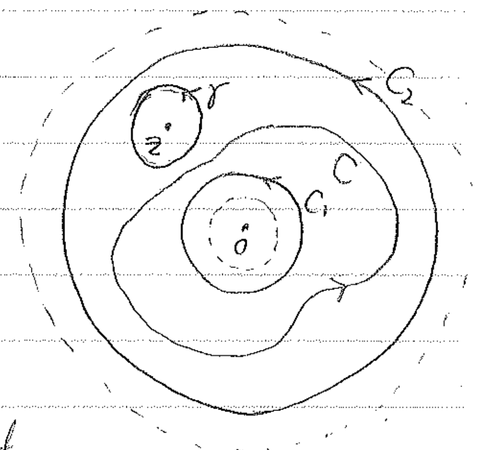
$$f(z) = \sum_{n=-\infty}^{+\infty} C_n (z-z_0)^n \quad \text{where } C_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

And this series is called a Laurant Series.

Remark. By our previous discussion, we know that if f is analytic throughout $|z-z_0| < R_2$, then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$, so in this case the Laurant series for f on $R_1 < |z-z_0| < R_2$ has no negative term, and this agrees with the expression $C_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ that when $n<0$, the integrand is analytic on and inside C in this case, so Cauchy-Goursat implies $C_n = 0$.

Proof. Similar to the proof for Taylor's Theorem, we'll first prove the case $z_0 = 0$.

Given a point z and a curve C on the domain $R_1 < |z - z_0| < R_2$, take $r_1, r_2 > 0$ such that $R_1 < r_1 < r_2 < R_2$ and z, C lie on ~~the~~ the smaller annulus $r_1 < |z| < r_2$. We denote C_1 and C_2 for the positively oriented circles $|z| = r_1$ and $|z| = r_2$



Take γ to be a small circle centred at z such that γ is in the smaller annulus $r_1 < |z| < r_2$ and γ is disjoint from C.

We apply the generalized Cauchy-Goursat Theorem to the multiply-connected domain inside C_2 and outside C_1 & γ, so we get

$$\int_{C_2} \frac{f(s)}{s-z}\,ds - \int_{C_1} \frac{f(s)}{s-z}\,ds - \int_{\gamma} \frac{f(s)}{s-z}\,ds = 0.$$

So $$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}\,dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}\,dz$$

Recall that in the proof of the Taylor's Theorem, we get

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + z^N \cdot \frac{1}{(s-z)s^N} \quad \text{for} \quad |z| < |s|.$$

So if we switch their roles, we get

$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + s^N \cdot \frac{1}{(z-s)z^N} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \cdot \frac{1}{z^{n+1}} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}$$

$$= \sum_{n=1}^{N} \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}$$

We thus obtain

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\left(\sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + z^N\cdot\frac{1}{(s-z)s^N}\right)f(s)\,ds$$

$$+\frac{1}{2\pi i}\int_{C_1}\left(\sum_{n=1}^{N}\frac{1}{s^{-n+1}}\cdot\frac{1}{z^n} + \frac{1}{z^N}\cdot\frac{s^N}{z-s}\right)f(s)\,ds$$

$$=\left(\sum_{n=0}^{N-1}\frac{z^n}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right) + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}\,ds$$

$$+\left(\sum_{n=1}^{N}\frac{1}{z^n}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right) + \frac{1}{z^N}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{f(s)\cdot s^N}{z-s}\,ds$$

$$=\sum_{n=0}^{N-1}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right)z^n + \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right)\cdot\frac{1}{z^n}$$

$$+ R_N$$

where $R_N = \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}\,ds + \frac{1}{z^N}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{f(s)s^N}{z-s}\,ds$.

and it's not hard to show $\lim_{N\to\infty} R_N = 0$.

so $$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}\,ds\right)z^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}\,ds\right)\frac{1}{z^n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{n+1}}\,ds\right)z^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{-n+1}}\,ds\right)\cdot\frac{1}{z^n}$$

Next for the general case. let $g(z) = f(z+z_0)$

f analytic in $R_1 < |z-z_0| < R_2 \Rightarrow g$ analytic in $R_1 < |z| < R_2$.

so for the simple closed contour C in $R_1 < |z-z_0| < R_2$.

if it's given by $z(t)$, $(a \le t \le b)$. then Γ be the contour $z(t) - z_0$. then Γ is in $R_1 < |z| < R_2$.

so by the special case at 0.

$$g(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma}\frac{g(s)}{s^{n+1}}ds\right)z^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma}\frac{g(s)}{s^{-n+1}}ds\right)\cdot\frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{(s-z_0)}ds\right)(z-z_0)^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{-n+1}}ds\right)\frac{1}{(z-z_0)^n}$$

Example. $f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z}\cdot\frac{1}{1+z^2}$ is analytic on $0<|z|<1$.

(The singularities are $0, \pm i$)

so on $0<|z|<1$,

$$f(z) = \frac{1}{z}\cdot\frac{1}{1+z^2} = \frac{1}{z}\left(\sum_{n=0}^{\infty}(-z^2)^n\right) = \frac{1}{z}\cdot\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^{2n-1}$$

Example. $f(z) = e^{\frac{1}{z}}$ is analytic on $0<|z|<\infty$, so on this domain

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}$$

Example. $f(z) = \frac{z+1}{z-1}$ is analytic on $0<|z-1|<\infty$. so on this domain:

$$f(z) = \frac{z+1}{z-1} = \frac{1}{z-1} + 2$$

Also, $f(z)$ is analytic on $1<|z|<\infty$. so

$$f(z) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=1}^{\infty}\frac{1}{z^n}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{2}{z^n}$$