Lemma. $\sum_{n=0}^{\infty} C_n$ is a convergent real series such that $C_n \geq 0 \; \forall n \in \mathbb{N}$.
If $\sum_{n=0}^{\infty} f_n(z)$ is a series of complex functions such that $|f_n(z)| < C_n \; \forall n \in \mathbb{N}$ on a region R, then $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on R.

Proof. $\sum_{n=0}^{\infty} C_n$ converges, so for $\forall \varepsilon > 0$. $\exists N_\varepsilon \in \mathbb{N}$ such that

$$N > N_\varepsilon \Rightarrow \left|\sum_{n=N+1}^{\infty} C_n\right| = \sum_{n=N+1}^{\infty} C_n < \varepsilon \quad (\text{since } C_n \geq 0 \; \forall n \in \mathbb{N})$$

Since $\forall z \in R$, $|f_n(z)| \leq C_n$, and $\sum_{n=1}^{\infty} C_n$ Converges, so by Comparison Test $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely to some $S(z)$, i.e there is a function $S(z)$ on R such that $S(z) = \sum_{n=0}^{\infty} f_n(z)$.

now for $N > N_\varepsilon \Rightarrow$, $$\left|S(z) - \sum_{n=0}^{N} f_n(z)\right| = \left|\sum_{n=N+1}^{\infty} f_n(z)\right| \leq \sum_{n=N+1}^{\infty} |f_n(z)| \leq \sum_{n=N+1}^{\infty} C_n < \varepsilon$$

so $\sum_{n=0}^{\infty} f_n(z)$ converges to $S(z)$ uniformly.

Theorem. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to a continuous function $S(z)$ inside its circle of convergence.

Proof. $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

If z_1 is inside the circle of convergence, take a circle C centred at z_0 such that z_1 is inside C and C is inside the circle of convergence. Then by the previous theorem, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges uniformly on the closed disk bounded by C.

For any $\varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ such that $N > N_\varepsilon \Rightarrow$

$$|S(z) - \sum_{n=0}^{N} a_n (z-z_0)^n| < \frac{\varepsilon}{3} \quad \forall \ z \text{ on or inside } C.$$

Note $\sum_{n=0}^{N} a_n (z-z_0)^n$ is a polynomial, so it's continuous at z_1, $\exists \delta > 0$ such that $|z - z_1| < \delta \Rightarrow$

$$\left| \sum_{n=0}^{N} a_n (z-z_0)^n - \sum_{n=0}^{N} a_n (z_1 - z_0)^n \right| < \frac{\varepsilon}{3}.$$

Take $0 < \delta' < \delta$ such that $|z - z_1| < \delta' \Rightarrow z$ is inside C.

$$\begin{aligned} |S(z) - S(z_1)| &\le |S(z) - \sum_{n=0}^{N} a_n (z-z_0)^n| + \left|\sum_{n=0}^{N} a_n (z-z_0)^n - \sum_{n=0}^{N} a_n (z_1 - z_0)^n\right| \\ &\quad + \left|\sum_{n=0}^{N} a_n (z_1 - z_0)^n - S(z_1)\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

so $S(z)$ is continuous at z_1

Lemma. C is a contour interior to the circle of convergence of $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, and $g(z)$ is a function that is continuous on C. Then

$$\int_C g(z) S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z-z_0)^n \, dz$$

Proof. Write $S(z) = \sum_{n=0}^{N-1} a_n (z-z_0)^n + \rho_N(z)$

$$g(z) S(z) = \sum_{n=0}^{N-1} a_n g(z) (z-z_0)^n + g(z) \rho_N(z)$$

Note since $S(z)$ & $\sum_{n=1}^{N-1} a_n (z-z_0)^n$ are continuous on C, $\rho_N(z)$ is also continuous on C.

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz$$

In order to prove the Lemma, we only need to show $\lim_{N\to\infty}\int_C g(z)\rho_N(z)\,dz = 0$.

Since $g(z)$ is continuous on C, let $M = \max_{z\in C}|g(z)|$

By the uniformly convergence of $S(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$,

$\forall\ \varepsilon>0,\ \exists\ N_\varepsilon>0$ such that

$$N>N_\varepsilon \Rightarrow \quad |\rho_N(z)| = \left|S(z)-\sum_{n=1}^{N-1} a_n(z-z_0)^n\right| < \frac{\varepsilon}{ML}\ \forall\ z\in C.$$

where L is the arclength of C.

So $N>N_\varepsilon \Rightarrow \left|\int_C g(z)\rho_N(z)\,dz\right| < M\cdot\frac{\varepsilon}{ML}\cdot L = \varepsilon$

we hence proved the Lemma.

Corollary. $S(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic inside the circle of convergence.

Proof. Pf. We take the constant function $g(z)\equiv 1$ in the previous Lemma, then for any closed contour C inside the circle of convergence,

$$\int_C S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,dz$$

Each term in the right side series is zero since $(z-z_0)^n$ is analytic $\forall\ n\in\mathbb{N}$.

We thus conclude $\int_C S(z)\,dz = 0$ for any closed contour C inside the circle of convergence, so $S(z)$ is analytic inside the circle of convergence.

Example. $f(z) = \begin{cases} \frac{\sin z}{z} & , z \neq 0 \\ 1 & , z = 0 \end{cases}$. We'll show $f(z)$ is entire:

so we need to show $f(z)$ is analytic at $z_0 = 0$

We know $\sin z = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

so for any $z \neq 0$,

$$f(z) = \frac{\sin z}{z} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{z^{2k}}{(2k+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

But $f(0) = 1$ satisfies the equation as well, so

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \quad \text{holds for all } z \in \mathbb{C}$$

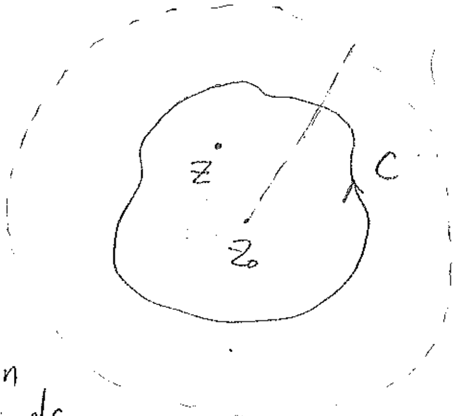
i.e. $f(z)$ equals to a series convergent everywhere, so $f(z)$ is entire (the circle of convergence has ∞-radius)

Theorem. $S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. For any z inside the circle of convergence,

$$S'(z) = \sum_{n=1}^{\infty} a_n n (z-z_0)^{n-1}$$

Proof. If z is inside the circle of convergence, take a positively oriented simple closed contour C such that C is inside the circle of convergence and z is inside C.

Let $g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s-z)^2}$ for $s \in C$



By the previous Lemma.

$$\int_C g(s)\, S(s)\, ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s-z_0)^n\, ds$$

$$\frac{1}{2\pi i}\int_C \frac{S(s)}{(s-z)^2}\, ds = \sum_{n=0}^{\infty} a_n \cdot \frac{1}{2\pi i}\int_C \frac{(s-z_0)^n}{(s-z)^2}\, ds$$

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{ds}(s-z_0)^n \Big|_{s=z}$$

$$S'(z) = \sum_{n=1}^{\infty} a_n \cdot n (z-z_0)^{n-1}$$

Example. We know $\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (|z-1|<1)$

so $-\frac{1}{z^2} = \left(\frac{1}{z}\right)' = \left(\sum_{n=0}^{\infty} (-1)^n (z-1)^n\right)'$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1}$$

so $\frac{1}{z^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n (z-1)^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n \quad (|z-1|<1)$

Theorem. If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for all points interior to some circle $|z-z_0| = R$, then $a_n = \frac{f^{(n)}(z_0)}{n!}$ i.e. the Taylor series expansion for an analytic function at z_0 is the unique series converging to f.

Proof. If $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ $(|z-z_0|<R)$, take C be a circle centred at z_0 with radius less than R.

Recall that we have proved $\int_C g(z)f(z)\,dz = \sum_{n=0} a_n \int_C g(z)(z-z_0)^n\,dz$

for any $g(z)$ continuous on C.

Now we choose $g_k(z) = \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^{k+1}}$, $k\in\mathbb{N}$ as $g(z)$.

so

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{n=0} a_n\cdot\frac{1}{2\pi i}\int_C (z-z_0)^{n-k-1}\,dz$$

$$\frac{f^{(k)}(z_0)}{k!} = a_k \quad \left(\text{since } \int_C (z-z_0)^{n-k-1}\,dz = \begin{cases} 0, & n\neq k \\ 2\pi i, & n=k \end{cases}\right)$$

Corollary If $\sum_{n=0}^{\infty} a_n(z-z_0)^n=0$ for any point inside a circle $|z-z_0|=R$, then $a_n=0$ $\forall\, n\in\mathbb{N}$