**Corollary.** $f$ is analytic at $z_0$, then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ in some neighbourhood of $z_0$

**Corollary.** $f$ is an entire function. $z_0 \in \mathbb{C}$. then for any $z \in \mathbb{C}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

**Example.** $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$$

**Example.** We have shown $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z|<1$.

Making use of this we can compute the Taylor series for some other functions.

First, replace $z$ by $-z$:

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{for } |z|<1$$

Next, replace $z$ by $1-z$. we get

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad \text{for } |z-1|<1$$

which is the Taylor expansion of $\frac{1}{z}$ at $z_0=1$

Now we consider the Taylor expansion of $\frac{1}{1-z}$ at $z_0 = i$.

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}}$$

$$= \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}}(z-i)^n \quad \text{for } |z-i|<\sqrt{2}$$

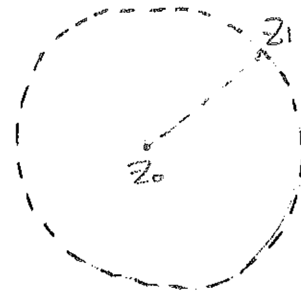
**Example.** $z^3 e^{2z} = z^3 \sum_{n=0}^{\infty} \frac{1}{n!}(2z)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{n+3}$

**Remark.** In the above examples, we indeed implicitly used the fact that the power series representation is unique. We'll show this fact later.

Now we are going to study some properties of power series.

**Theorem.** If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$, then it is absolutely converged at any point on the open disk $|z - z_0| < |z_1 - z_0|$.

**Proof.** If $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges,




$\lim_{n\to\infty} a_n(z_1-z_0)^n = 0$. So

there exists $M > 0$ such that

$|a_n(z_1-z_0)^n| \leq M \quad \forall\, n \in \mathbb{N}$.

Then for any $z$ such that $|z-z_0| < |z_1 - z_0|$,

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n \leq M \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n$$

Since $\left|\frac{z-z_0}{z_1-z_0}\right| < 1$, the series $\sum_{n=0}^{\infty} M \cdot \left|\frac{z-z_0}{z_1-z_0}\right|^n$ converges.

By Comparison Test, this indicates $\sum_{n=0}^{\infty} |a_n(z-z_0)|^n$ converges, i.e. $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent.

**Definition.** The greatest circle centred at $z_0$ such that $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is convergent at every point inside is called the circle of convergence of the series.

**Corollary** If a circle $C$ is the circle of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, then the series diverges at any point outside of $C$.

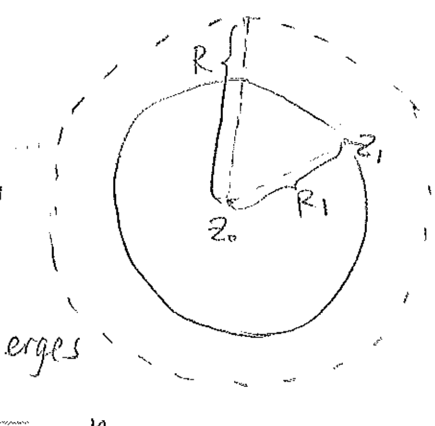
**Proof.** If the power series converges at some $z$ outside of $C$, then by the theorem, the series is absolutely convergent inside a circle of radius $|z-z_0|$, larger than that of $C$, contradiction.

**Definition.** A series $\sum_{i=0}^{\infty} f_n(z)$ converges uniformly to a function $S(z)$ on a region $R$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$N > N_0 \Rightarrow \quad \left|\sum_{i=0}^{N} f_n(z) - S(z)\right| < \varepsilon \quad \forall z \in R$$

**Theorem** If $\sum_{i=0}^{\infty} a_n (z-z_0)^n$ has circle of convergence $|z-z_0| = R$, $0 < R_1 < R$, then the series converges uniformly on the closed disk $|z-z_0| \le R_1$

**Proof.** Take a point $z_1$ such that $|z_1 - z_0| = R_1$. Since $z_1$ is inside the circle of convergence, the power series converges absolutely at $z_1$, i.e.




$$\sum_{n=0}^{\infty} |a_n (z_1 - z_0)|^n = \sum_{n=0}^{\infty} |a_n R_1|^n \text{ converges}$$

For any $|z - z_0| \le R'$, $|a_n (z-z_0)|^n \le |a_n R_1|^n \quad \forall n \in \mathbb{N}$.

So we'll finish the proof by directly applying the following Lemma.