SERIES

Definition. An infinite sequence $\{z_n\}$ of complex numbers has limit z if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |z_n - z| < \varepsilon.$$

We say $\{z_n\}$ converges to z if $\{z_n\}$ has limit z, and denoted by $\lim_{n\to\infty} z_n = z$.

We can study the convergence of a sequence by studying its real and imaginary parts.

Theorem. $z_n = x_n + iy_n$ is a sequence and $z = x + iy$ is a complex number. Then

$$\lim_{n\to\infty} z_n = z \text{ if and only if } \lim_{n\to\infty} x_n = x \ \& \ \lim_{n\to\infty} y_n = y.$$

Proof "$\Leftarrow$" If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$,

then for any $\varepsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ & $N_2 \in \mathbb{N}$ such that $n > N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$,

$n > N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2}$

So for any $n > \max\{N_1, N_2\}$

$$|z_n - z| \le |x - x_n| + |y - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $\lim_{n\to\infty} z_n = z$

"$\Rightarrow$" If $\lim_{n\to\infty} z_n = z$

then for any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $n > N \Rightarrow |z_n - z| < \varepsilon$

For any $n > N$

$|x_n - x| \le |z_n - z| < \varepsilon$ and $|y_n - y| \le |z_n - z| < \varepsilon$

so $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Example. $z_n = -1 + i \cdot \frac{(-1)^n}{n^2}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-1) + i \lim_{n\to\infty} \frac{(-1)^n}{n^2} = -1 + i \cdot 0 = -1$$

You may also use definition to conclude $\lim_{n\to\infty} z_n = -1$:

$\forall \varepsilon > 0$, let $N > \frac{1}{\sqrt{\varepsilon}}$,

$$n > N \Rightarrow |z_n - (-1)| = \left| i \cdot \frac{(-1)^n}{n^2} \right| = \frac{1}{n^2} < \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2} = \varepsilon$$

Remark. z_n converges to z doesn't necessarily imply $\mathrm{Arg}(z_n)$ converges to $\mathrm{Arg}(z)$.

We may take the above example again
It's easy to see

$$\lim_{k\to\infty} \mathrm{Arg}(z_{2k-1}) = -\pi$$

but $\lim_{k\to\infty} \mathrm{Arg}(z_{2k}) = \pi$

so $\mathrm{Arg}(z_n)$ is not convergent

Exercise. Prove that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} |z_n| = |z|$

Definition. An infinite series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$ of complex numbers converges to the sum S if the sequence of partial sums $S_N = \sum_{n=1}^{N} S_n$ converges to S.

We write $\sum_{i=1}^{\infty} z_n = S$

Theorem. $z_n = x_n + iy_n$ is a sequence and $S = X + iY$ is a complex number. Then

$$\sum_{n=1}^{\infty} z_n = S \text{ if and only if } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

Proof. Let $S_N = \sum_{n=1}^{N} z_n$, $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$

Then $S_N = X_N + iY_N$

So $\lim_{N\to\infty} S_n = S$ if and only if $\lim_{N\to\infty} X_N = X$ & $\lim_{N\to\infty} Y_N = Y$

i.e. $\sum_{i=1}^{\infty} z_n = S$ if and only if $\sum_{i=1}^{\infty} x_n = X$ & $\sum_{i=1}^{\infty} y_n = Y$

Corollary. If $\sum_{i=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Proof Write $z_n = x_n + iy_n$. If $\sum_{i=1}^{\infty} z_n$ converges

$\sum_{i=1}^{\infty} z_n = \sum_{i=1}^{\infty} x_n + i \cdot \sum_{i=1}^{\infty} y_n$, $\sum_{i=1}^{\infty} x_n$ & $\sum_{i=1}^{\infty} y_n$ are convergent.

So $\lim_{n\to\infty} x_n = 0$ & $\lim_{n\to\infty} y_n = 0$

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0.$

Definition. A series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Proposition. An absolutely convergent series is convergent.

Proof. If $\sum_{n=1}^{\infty} |z_n|$ is convergent, then by the Comparison Test,

$\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ are convergent since

$|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$ for all n.

We see $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent since they're absolutely convergent. so we conclude

$\sum_{n=1}^{\infty} z_n$ is convergent

Definition A power series is a series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$. where $a_n \in \mathbb{C}$, $z_0 \in \mathbb{C}$ and z is the variable.

Example $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for any $|z|<1$.

$S_N = \sum_{n=1}^{N} z^N$. define $\rho_N = \frac{1}{1-z} - \sum_{n=1}^{N} z^N$

To show $\lim_{N\to\infty} S_N = \frac{1}{1-z}$ is the same as to show

$\lim_{N\to\infty} \rho_N = 0$

$S_N = \sum_{n=1}^{N} z^N = \frac{1-z^{N+1}}{1-z}$

$$\rho_N(z) = \frac{1}{1-z} - \frac{1-z^{N+1}}{1-z} = \frac{z^{N+1}}{1-z}$$

we see for each fixed $|z|<1$,

$$|\rho_N(z)| = \frac{|z|^{N+1}}{|1-z|} \to 0 \quad \text{as } N \to \infty$$

so $\lim\limits_{N\to\infty} \rho_N(z) = 0$, $\lim\limits_{n\to\infty} S_N = \frac{1}{1-z}$. i.e $\sum\limits_{n=1}^{\infty} z^n = \frac{1}{1-z}$

Observation. In the example above, we proved

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } |z|<1.$$

Let $f(z) = \frac{1}{1-z}$ defined in the open disk $|z|<1$,
$f(z)$ is analytic on this disk.

$$f'(z) = \frac{1}{(1-z)^2}, \quad f''(z) = \frac{2}{(1-z)^3}, \quad \dots, \quad f^{(k)}(z) = \frac{k!}{(1-z)^{k+1}}, \dots$$

So $f'(0) = 1$, $f''(0) = 2$, ..., $f^{(k)}(0) = k!$, ...

The equation tells us

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^2$$ on the disk centred

at 0 with radius 1.

Definition. f is a function that is analytic at z_0, then define the Taylor Series of f at z_0 to be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

TAYLOR SERIES

In the previous example, we see for the analytic function $f(z) = \frac{1}{1-z}$ on the disk $|z|<1$, $f(z)$ equals its Talyor Series at $z_0 = 0$. Now we would like to see in general if there is such kind of results.

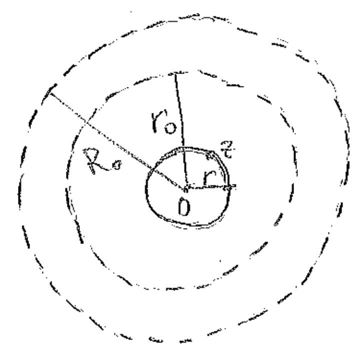
Theorem. f is analytic throughout a disk $|z-z_0|<R_0$ (centred at z_0 and with radius R_0). Then:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{for all } |z-z_0|<R_0$$

Proof. We will first show the case $z_0=0$, then obtain the general case.

For any $|z|<R_0$, denote $r=|z|$.

Pick $r<r_0<R_0$, and denote the circle $|z|=r_0$ to be C_0, positively oriented.



f is analytic on and inside C_0, so Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\,ds$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s} \cdot \frac{1}{1-\frac{z}{s}}\,ds$$

Recall that in the previous example, we have shown that for any $|z|<1$, $\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$

So $\frac{1}{1-\frac{z}{s}} = \sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\left(\frac{z}{s}\right)} = \sum_{n=0}^{N-1}\frac{z^n}{s^n} + \frac{z^N s}{s^N(s-z)}$

So $f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s}\cdot\frac{1}{1-\frac{z}{s}}\,ds$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s}\left(\sum_{n=0}^{N-1}\frac{z^n}{s^n} + \frac{z^N s}{s^N(s-z)}\right)ds$$

$$= \sum_{n=0}^{N-1}\frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{s^{n+1}}\,ds\cdot z^n + \frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{s^N(s-z)}\,ds\cdot z^N$$

$$= \sum_{n=0}^{N-1}\frac{f^{(n)}(0)}{n!}z^n + \frac{z^N}{2\pi i}\int_{C_0}\frac{f(s)}{s^N(s-z)}\,ds$$

We need to show $\lim_{N\to\infty}\frac{z^N}{2\pi i}\int_{C_0}\frac{f(s)}{s^N(s-z)}\,ds = 0$

Let $M = \max_{s\in C_0} f(s)$. For any $|s| = r_0$ and any $|z| = r$,

$|s-z| \geq ||s|-|z|| = r_0 - r$.

So $\left|\frac{z^N}{2\pi i}\int_{C_0}\frac{f(s)}{s^N(s-z)}\,ds\right| \leq \frac{r^N}{2\pi}\cdot\frac{M}{r_0^N\cdot(r_0-r)}\cdot 2\pi r_0 = \frac{Mr_0}{r_0-r}\cdot\left(\frac{r}{r_0}\right)^N \to 0$

as $N\to\infty$ we get $\lim_{N\to\infty}\frac{z^N}{2\pi i}\int_{C_0}\frac{f(s)}{s^N(s-z)}\,ds = 0$

So $f(z) = \lim_{N\to\infty}\sum_{n=0}^{N-1}\frac{f^{(n)}(0)}{n!}z^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$.

Next, we prove the general case.

Let $g(z) = f(z+z_0)$, then $g(z)$ is analytic inside the circle $|z| = R_0$. By the $z_0 = 0$ case just proved,

$$g(z) = \sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}z^n$$

$$\Rightarrow f(z+z_0) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}z^n$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$