Homework II Solution

First-Half

1. The tangent vector is $\overrightarrow{r}'(t) =< -\sin t, 3, 4\cos 2t >$, so the tangent vector at $t = 0$ is $\overrightarrow{r}'(0) =< 0, 3, 4 >$, and the unit tangent vector is $\frac{\overrightarrow{r}'(0)}{|\overrightarrow{r}'(0)|} =< 0, \frac{3}{5}, \frac{4}{5} >$

2. The tangent vector is $\overrightarrow{r}'(t) =< -2\sin t, 2\cos t, e^t >$. it's parallel to the plane $\sqrt{3}x + y = 1$ if and only if it is perpendicular to the normal vector of the plane, which is $< \sqrt{3}, 1, 0 >$. So we solve for

$$< -2\sin t, 2\cos t, e^t > \cdot < \sqrt{3}, 1, 0 >= -2\sqrt{3}\sin t + 2\cos t = 0$$

which implies $\tan t = \frac{\sin t}{\cos t} = \frac{1}{\sqrt{3}}$, and $0 \le t \le \pi$, so $t = \frac{\pi}{6}$

$$\vec{r}(\frac{\pi}{6}) =< 2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}, e^{\frac{\pi}{6}} >=< \sqrt{3}, 1, e^{\frac{\pi}{6}} >$$

3. Let $\vec{u}(t) =< f(t), g(t), h(t) >$ and $\vec{v}(t) =< \phi(t), \psi(t), \zeta(t) >$, then

$$\vec{u}(t)\times\vec{v}(t) =< g(t)\zeta(t)-h(t)\psi(t), h(t)\phi(t)-f(t)\zeta(t), f(t)\psi(t)-g(t)\phi(t) >$$

$$\begin{aligned}(\vec{u}(t) \times \vec{v}(t))' =<&g'(t)\zeta(t) + g(t)\zeta'(t) - h'(t)\psi(t) - h(t)\psi'(t),\\ &h'(t)\phi(t) + h(t)\phi'(t) - f'(t)\zeta(t) - f(t)\zeta'(t),\\ &f'(t)\psi(t) + f(t)\psi'(t) - g'(t)\phi(t) - g(t)\phi'(t) >\end{aligned}$$

$$\overrightarrow{u}'(t)\times\vec{v}(t) =< g'(t)\zeta(t)-h'(t)\psi(t), h'(t)\phi(t)-f'(t)\zeta(t), f'(t)\psi(t)-g'(t)\phi(t) >$$
$$\vec{u}(t)\times\overrightarrow{v}'(t) =< g(t)\zeta'(t)-h(t)\psi'(t), h(t)\phi'(t)-f(t)\zeta'(t), f(t)\psi'(t)-g(t)\phi'(t) >$$

$$\begin{aligned}\overrightarrow{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\overrightarrow{v}'(t) = <&g'(t)\zeta(t)+g(t)\zeta'(t)-h'(t)\psi(t)-h(t)\psi'(t),\\ &h'(t)\phi(t)+h(t)\phi'(t)-f'(t)\zeta(t)-f(t)\zeta'(t),\\ &f'(t)\psi(t)+f(t)\psi'(t)-g'(t)\phi(t)-g(t)\phi'(t)>\end{aligned}$$

So we see $(\vec{u}(t)\times\vec{v}(t))' = \overrightarrow{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\overrightarrow{v}'(t)$

4. $\overrightarrow{r}'(t) =< 1, -3\sin t, 3\cos t >$, so $|\overrightarrow{r}'(t)| = \sqrt{1^2+(-3\sin t)^2+(3\cos t)^2} = \sqrt{10}$.

$$\int_0^\pi \sqrt{10}dt = \sqrt{10}\pi$$

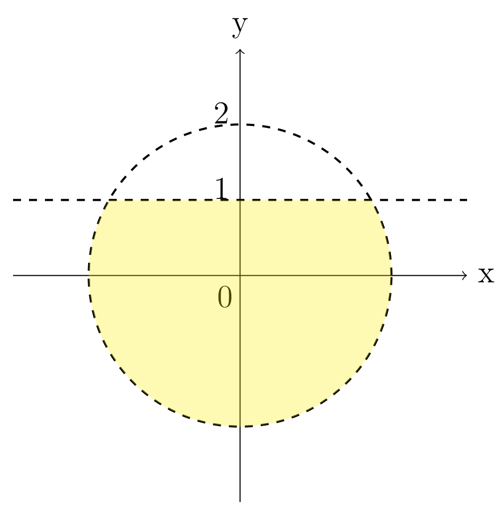
5. Let $f(t) = |\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$. Then since $\vec{r}(t)\perp\overrightarrow{r}'(t)$,

$$f'(t) = \overrightarrow{v}'(t)\cdot\vec{v}(t)+\vec{v}(t)\cdot\overrightarrow{v}'(t) = 0$$

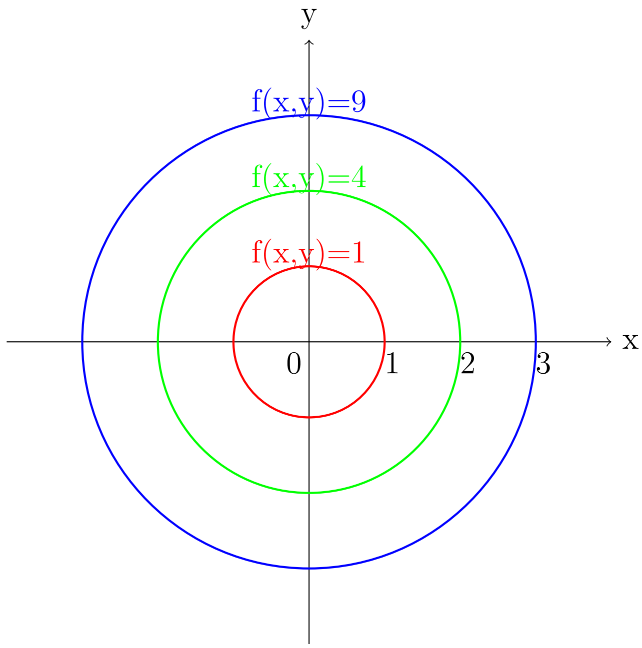
So $f(t)\equiv C$ for some $C\in\mathbb{R}$, i.e. $|\vec{r}(t)|^2 = C$ for all t, we conclude $\vec{r}(t)$ lies on the sphere of radius $\sqrt{C}$ centered at origin.

6. We need $4-x^2-y^2>0$ and $1-y>0$, i.e. $x^2+y^2<4$ and $y<1$. So the domain is

$$\{(x,y)\in\mathbb{R}^2 | x^2+y^2<4 \text{ and } y<1\}$$



7.



8. If $f(x, y)$ approaches $(0, 0)$ along positive x direction:

$$f(x, 0) = \frac{x^4}{x^2} = x^2$$

So $f(x, 0) \to 0$ as $x \to 0^+$

If $f(x, y)$ approaches $(0, 0)$ along positive y direction:

$$f(x, 0) = \frac{-4y^2}{2y^2} = -2$$

So $f(0, y) \to -2$ as $y \to 0^+$.

We conclude the limit doesn't exist

9. If $f(x, y)$ approaches $(0, 0)$ along positive x direction:

$$f(0, y) = \frac{y}{y} = 1$$

So $f(0, y) \to 1$ as $y \to 0^+$

If $f(x, y)$ approaches $(0, 0)$ along positive y direction:

$$f(x, 0) = \frac{x^a}{x} = x^{a-1}$$

So as $x \to 0^+$, $f(x, 0) \to 0$ if $a > 1$ and $f(x, 0) \to +\infty$ if $a < 1$.

We conclude the limit doesn't exist.

10. For any $\epsilon > 0$, we can take $\delta = \epsilon$, then:
    For any $0 < \sqrt{x^2 + y^2} < \delta = \epsilon$,

$$\begin{aligned}
|f(x, y) - 0| &= |\frac{xy}{\sqrt{x^2 + y^2}}| \\
&= \frac{|x||y|}{\sqrt{x^2 + y^2}} \\
&\leq \frac{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \\
&= \sqrt{x^2 + y^2} \\
&< \delta \\
&= \epsilon
\end{aligned}$$

So $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$

Second-Half

1. (a). $\frac{\partial f}{\partial x} = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$

   $\frac{\partial f}{\partial y} = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$

   So $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{2y}{(x+y)^2}) = (2y)(-2(x+y)^{-3}) = \frac{-4y}{(x+y)^3}$

   $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\frac{-2x}{(x+y)^2}) = -2\frac{(x+y)^2 - x(2(x+y))}{(x+y)^4} = \frac{-2(y^2-x^2)}{(x+y)^4} = \frac{2(x-y)}{(x+y)^3}$

   $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(\frac{2y}{(x+y)^2}) = 2\frac{(x+y)^2 - y(2(x+y))}{(x+y)^4} = \frac{2(x^2-y^2)}{(x+y)^4} = \frac{2(x-y)}{(x+y)^3}$

   $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{-2x}{(x+y)^2}) = (-2x)(-2(x+y)^{-3}) = \frac{4x}{(x+y)^3}$

   (b). $\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+y^2}}$

   $\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2+y^2}}$

   $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{x}{\sqrt{x^2+y^2}}) = \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\frac{x^2+y^2-x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$

   $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\frac{y}{\sqrt{x^2+y^2}}) = y(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}})(2x) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

   $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(\frac{x}{\sqrt{x^2+y^2}}) = x(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}})(2y) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

   $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{y}{\sqrt{x^2+y^2}}) = \frac{\sqrt{x^2+y^2} - y\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\frac{x^2+y^2-y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$

   (c). $\frac{\partial f}{\partial x} = \frac{y}{x}$, $\frac{\partial f}{\partial y} = \ln x$

   $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{y}{x}) = -\frac{y}{x^2}$

   $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$

   $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(\frac{y}{x}) = \frac{1}{x}$

   $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\ln x) = 0$

2. $\frac{\partial w}{\partial x} = 3yz + 2xy - z^3$, $\frac{\partial w}{\partial y} = 3xz + x^2$, $\frac{\partial w}{\partial z} = 3xy - 3xz^2$

$\frac{\partial^2 w}{\partial x^2} = 2y$	$\frac{\partial^2 w}{\partial x \partial y} = 3z + 2x$	$\frac{\partial^2 w}{\partial x \partial z} = 3y - 3z^2$
$\frac{\partial^2 w}{\partial y \partial x} = 3z + 2x$	$\frac{\partial^2 w}{\partial y^2} = 0$	$\frac{\partial^2 w}{\partial y \partial z} = 3x$
$\frac{\partial^2 w}{\partial z \partial x} = 3y - 3z^2$	$\frac{\partial^2 w}{\partial z \partial y} = 3x$	$\frac{\partial^2 w}{\partial z^2} = -6xz$

3.

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (4x)(2t) + 6y \\ &= 8(t^2 - s)t + 6(t + 2s^3) \\ &= 8t^3 - 8st + 6t + 12s^3\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (4x)(-1) + (6y)(6s^2) \\ &= -4(t^2 - s) + 36(t + 2s^3)s^2 \\ &= -4t^2 + 4s + 36ts^2 + 72s^5\end{aligned}$$

4. $\frac{\partial z}{\partial x} = \frac{1}{2}\sqrt{\frac{y}{x}}$ and $\frac{\partial z}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}}$, so $\frac{\partial z}{\partial x}(1,1) = \frac{1}{2}$ and $\frac{\partial z}{\partial y}(1,1) = \frac{1}{2}$.

   So the equation of the tangent plane at $(1,1,1)$ is

$$z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

5. $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (3x^2 \ln y)dx + \frac{x^3}{y}dy$

6.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{6xy + 5z^2}{4y^2 + 10zx}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{3x^2 + 8yz}{4y^2 + 10zx}$$

7. Method I:

   $0 \approx dC = \frac{\partial C}{\partial x}(1,1)dx + \frac{\partial C}{\partial y}(1,1)dy = 6(1.1 - 1) + 10(y - 1)$

   So $y \approx 0.94$

   Method II:

   $\frac{dy}{dx} = -\frac{\frac{\partial C}{\partial x}}{\frac{\partial C}{\partial y}} = -\frac{3x^2y + 3y^3}{x^3 + 9xy^2}$.

When $(x, y) = (1, 1)$, $\frac{dy}{dx}(1) = -\frac{3}{5}$.
So $y \approx 1 - \frac{3}{5}(x-1) = 1 - \frac{3}{5}(1.1 - 1) = 0.94$

8. $\frac{dD}{dt} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{ds} = \frac{x}{\sqrt{x^2+y^2}}(2t) + \frac{y}{\sqrt{x^2+y^2}}(3t^2+1)$
When $t = 2$, $x = 2^2 = 4$ and $y = 2^3 + 2 = 10$, so
$V_r(2) = \frac{dD}{dt}(2) = \frac{4}{\sqrt{4^2+10^2}}(2 \times 2) + \frac{10}{\sqrt{4^2+10^2}}(3 \times 2^2 + 1) = \frac{146}{\sqrt{116}} = \frac{73}{\sqrt{29}} = \frac{73}{29}\sqrt{29}$

9. The profit function is

$$\pi(L) = pf(L) - wL$$

When profit is maximized at $L = L^*$, $\pi'(L^*) = 0$, i.e.

$$pf'(L^*) - w = 0$$

(And $f''(L) < 0$ implies $\pi''(L) = pf''(L) < 0$, so π is a concave function, hence obtains maximum at critical point)
Let $F(p, w, L^*) = pf'(L^*) - w$, then by implicit differentiation:

$$\begin{cases} \frac{\partial L^*}{\partial p} = -\frac{\frac{\partial F}{\partial p}}{\frac{\partial F}{\partial L^*}} = -\frac{f'(L^*)}{pf''(L^*)} > 0 \\ \\ \frac{\partial L^*}{\partial w} = -\frac{\frac{\partial F}{\partial w}}{\frac{\partial F}{\partial L^*}} = \frac{1}{pf''(L^*)} < 0 \end{cases}$$

We can conclude that if price increases, the optimal labor input goes up; if labor cost increases, the optimal labor input goes down.

10. $\frac{\partial f}{\partial x} = 2x + y\cos xy$, $\frac{\partial f}{\partial y} = x\cos xy$, so $\frac{\partial f}{\partial x}(1, 0) = 2$, $\frac{\partial f}{\partial y}(1, 0) = 1$
If the unit vector $\vec{u} =< a, b >$, then

$$D_{\vec{u}}f = 2a + b = 1$$

Together with $a^2 + b^2 = 1$, we get $a = 0, b = 1$ or $a = \frac{4}{5}, y = -\frac{3}{5}$. So $\vec{u} =< 0, 1 >$ or $\vec{u} =< \frac{4}{5}, -\frac{3}{5} >$

11. The fastest increase of f at (x,y) is along $\nabla f(x,y)$ direction.

$$\nabla f(x,y) = <\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> = <2x-2, 2y-4>$$

We need that $<2x-2, 2y-4>$ is in the same direction as $<1,1>$, so there exists positive $\lambda \in \mathbb{R}$ such that

$$<2x-2, 2y-4> = \lambda <1,1> = <\lambda, \lambda>$$

We get

$$x = \frac{\lambda}{2} + 1, y = \frac{\lambda}{2} + 2, \lambda > 0$$

i.e. The ray $x - y + 1 = 0$, $x > 1$

So f increases fastest along $<1,1>$ direction on the set of points $\{(\frac{\lambda}{2}+1, \frac{\lambda}{2}+2) | \lambda > 0\}$, which is the ray $x - y + 1 = 0$, $x > 1$

12. The gradient vector is $\nabla z = <\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}> = <-0.01x, -0.02y>$, at $(60, 40, 966)$ it is $\nabla z = <-0.6, -0.8>$

(a). The south direction has unit directional vector $\vec{u} = <0, -1>$, so

$$D_{\vec{u}} z = <-0.6, -0.8> \cdot <0, -1> = 0.8$$

So you will start to ascend at rate of 0.8

(b). The northwest direction has unit directional vector $\vec{u} = <-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}>$, so

$$D_{\vec{u}} z = <-0.6, -0.8> \cdot <-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}> = -\frac{\sqrt{2}}{10}$$

So you will start to ascend at rate of 0.8

(c). The slope is largest in the direction of the gradient $\nabla z = <-0.6, -0.8>$. The rate of ascend is $| <-0.6, -0.8> | = 1$. If we assume the angle is θ, then $\tan\theta = 1$, we get $\theta = \frac{\pi}{4}$.

13. Let $f(x,y,z) = x^2 - y^2 - z^2$, then this surface is the lever surface $f(x,y,z) = 1$. So at each point on the surface, the gradient vector $\nabla f = <2x, -2y, -2z>$ is normal to the tangent plane. If the tangent plane is parallel to $z = x + y$, then the corresponding normal vectors are parallel, i.e. $<2x, -2y, -2z> // <1, 1, -1>$. So there exists $\lambda \in \mathbb{R}$ such that $<2x, -2y, -2z> = \lambda <1, 1, -1>$

So we get

$$\begin{cases} 2x = \lambda \\ -2y = \lambda \\ -2z = -\lambda \\ x^2 - y^2 - z^2 = 1 \end{cases}$$

The first three equations tell us $x = \frac{\lambda}{2}, y = -\frac{\lambda}{2}, z = \frac{\lambda}{2}$, putting into the last equation, we see

$$(\frac{\lambda}{2})^2 - (\frac{-\lambda}{2})^2 - (\frac{\lambda}{2})^2 = 1$$

which implies

$$-(\frac{\lambda}{2})^2 = 1$$

We see this equation has no solution. So there is no tangent plane parallel to $z = x + y$.