Volume

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1 Method of Cross Section

We can generalise the method of computing areas to one dimension higher, to compute the volume of some given solid. Given a solid, in order to find its volume, the idea is to first fix a direction in space, and make parallel slices of the solid that are perpendicular to the fixed direction, then the volume should be computable using the Riemann sum of the volume of these slices.

Theorem 1. *If E is a solid whose projection to the z-axis is $[a, b]$, and the the area of the horizontal cross-section of E is $A(z)$ at height z, then the volume of E is*

$$\int_a^b A(z)\,dz$$

Proof. We divide the interval $[a, b]$ on z-axis into n pieces of equal length $\Delta z = \frac{b-a}{n}$, and consider the cylindrical slices with base $A(z_i)$ and height Δz. The sum of the volume of these slices is

$$\sum_{i=1}^{n} A(z_i)\Delta z$$

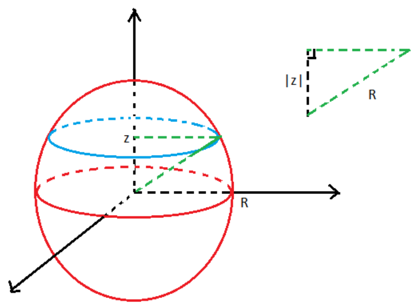
As n getting larger, the space covered by these slices will converge to E, so the volume of E is

$$\lim_{\Delta z \to 0} \sum_{i=1}^{n} A(z_i)\Delta z = \int_a^b A(z)\,dz$$

□

Remark 2. There are also analogues of the above theorem if we take the x-axis or y-axis instead of the z-axis. The choice of axis depends on the convenience of computation.

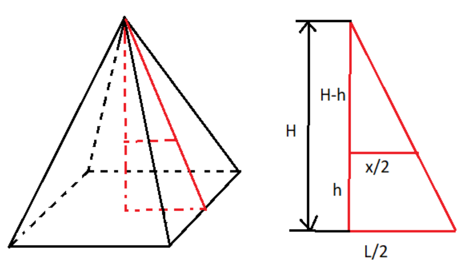
Example 3. *Compute the volume of a ball of radius R.*



We set the centre of the ball at the origin of the coordinates. Then at height z, the cross section is a disk of radius $\sqrt{R^2 - z^2}$, so the volume is

$$\int_{-1}^{1} \pi(R^2 - z^2)\, dz = \pi(R^2 z - \frac{z^3}{3})\Big|_{-R}^{R} = \frac{4}{3}\pi R^3$$

Example 4. *Compute the volume of a pyramid whose height is H and base is a square with each edge having length L.*



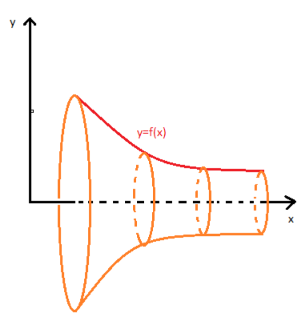
At height h, by similar triangles, we can find the length of each edge of *the section to be* $x = \frac{H-h}{H}L$. *So by the formula, the volume is*

$$\begin{aligned}\int_0^H (\frac{H-h}{H}L)^2\,dh &= \frac{L^2}{H^2}\int_0^H (H-h)^2\,dh\\ &= \frac{L^2}{H^2}\int_0^H (h-H)^2\,d(h-H)\\ &= \frac{L^2}{H^2}\frac{(h-H)^3}{3}\Big|_0^H\\ &= \frac{1}{3}HL^2\end{aligned}$$

A special case of the previous theorem is when the volume is generated from rotation of a curve:

Theorem 5. *Given the graph of* $y = f(x)$ *on the* xy*-plane with domain* $[a, b]$*, if we rotate the graph about* x*-axis in the* 3*-dimensional space, the graph sweeps over a surface. The volume bounded inside this surface is given by*

$$\pi\int_a^b f(x)^2\,dx$$



Proof. We compute along x-axis. The cross section for each x is a disk of radius $f(x)$, so the area of the cross section is $\pi f^2(x)$. Applying Theorem 1, we see the volume is

$$\int_a^b \pi f^2(x)\,dx = \pi\int_a^b f^2(x)\,dx$$

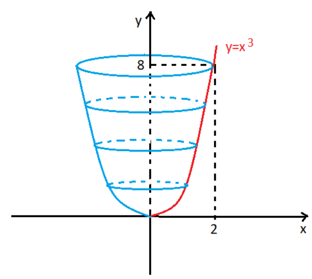
□

Example 6. *Compute the volume of the solid obtained from rotating $y = \frac{1}{x}$ along x-axis between $[1, 2]$.*

$$\pi \int_1^2 (\frac{1}{x})^2 \, dx = \pi \int_1^2 \frac{1}{x^2} \, dx = -\pi \frac{1}{x}\Big|_1^2 = \frac{\pi}{2}$$

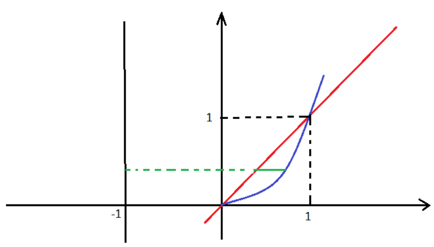
We can also apply the above method to volumes obtained from rotating along y-axis.

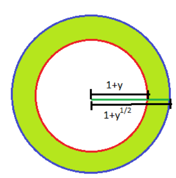
Example 7. *Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.*



$$\pi \int_0^8 (y^{\frac{1}{3}})^2 \, dy = \frac{3\pi}{5} y^{\frac{5}{3}}\Big|_0^8 = \frac{96\pi}{5}$$

Example 8. *Find the volume of the solid obtained by rotating the region on xy-plane enclosed by $y = x$ and $y = x^2$ about the line $x = -1$.*





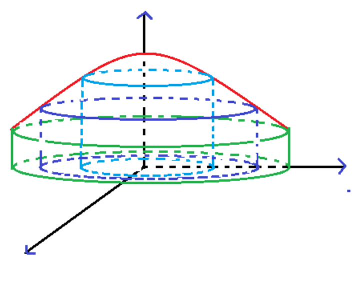
$$\int_0^1 \pi(\sqrt{y} - (-1))^2 - \pi(y - (-1))^2 \, dy = \pi \int_0^1 (\sqrt{y} + 1)^2 - (y + 1)^2 \, dx = \frac{\pi}{2}$$

2 Method of Cylindrical Shells

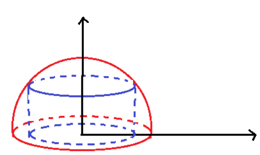
Another useful technique to compute volume is to use cylinders shells instead of cross sections.

Theorem 9. *$f(x)$ is a function defined on the interval $[a, b]$ $(0 \le a < b)$. E is the region swept by rotation the region under $z = f(x)$ around the z-axis. Then the volume of E is*

$$2\pi \int_a^b x f(x)\, dx$$



Example 10. *Let's compute the volume of the ball of radius R using cylindrical shells:*



We first compute the part above xy-plane. This part can be regarded as obtained from rotating the region under $z = \sqrt{R^2 - x^2}$ around z-axis. Then its volume is

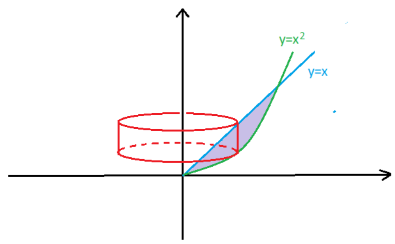
$$2\pi \int_0^R x\sqrt{R^2 - x^2}\, dx = -\pi \int_0^R \sqrt{R^2 - x^2}\, d(R^2 - x^2) = -\frac{2\pi}{3}(R^2 - x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{2\pi}{3}R^3$$

Then by symmetry, the volume of the ball should be twice of what we have just computed: $2 \times \frac{2\pi}{3}R^3 = \frac{4\pi}{3}R^3$

Example 11. *Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.*

The volume is

$$\int_0^1 2\pi x(x - x^2)\, dx = \frac{2\pi}{3}x^3 - \frac{\pi}{2}x^4 \Big|_0^1 = \frac{\pi}{6}$$



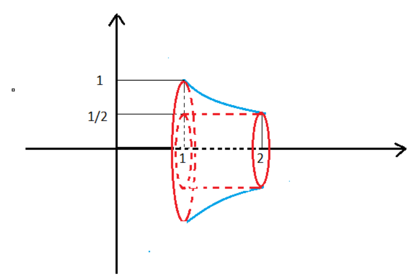
Example 12. *Find the volume of the solid obtained by rotating about the line $x = -1$ the region between $y = x$ and $y = x^2$.*

(Note that we have computed example before using the method of cross sections, see Example 8. You are encouraged to compare the two different methods.)

For the cylindrical shell at x, the radius of the base becomes $x + 1$, so the volume is:

$$\int_0^1 2\pi(x + 1)(x - x^2)\, dx = \frac{\pi}{2}$$

Example 13. *Find the volume of the solid obtained by rotating about x-axis the region below $y = \frac{1}{x}$ between $[1, 2]$. (Compare with Exaple 6)*



$$\int_{\frac{1}{2}}^1 2\pi y(\frac{1}{y} - 1)\, dy + \pi \times (\frac{1}{2})^2 \times (2 - 1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$