1. Find a power series representation for the function and determine the interval of convergence:

   (i). $f(x) = \frac{x}{9+x^2}$

   (ii). $f(x) = \frac{1+x}{1-x}$

   (iii). $f(x) = \frac{3}{x^2-x-2}$

   (iv). $f(x) = \frac{x}{(1+4x)^2}$

   Solution:

   (i).

$$\frac{x}{9+x^2} = \frac{x}{9} \times \frac{1}{1+\frac{x^2}{9}} = \frac{x}{9}\sum_{n=0}^{\infty}(-\frac{x^2}{9})^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{9^{n+1}}x^{2n+1}$$

   The series converges on $|\frac{x^2}{9}| < 1$, i.e., $-3 < x < 3$. The interval of convergence is $(-3, 3)$.

   (ii).

$$\frac{1+x}{1-x} = -1 + \frac{2}{1-x} = -1 + 2\sum_{0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

   The interval of convergence is $(-1, 1)$.

   (iii).

$$\frac{3}{x^2-x-2} = \frac{1}{x-2} = \frac{1}{x+1} = -\frac{1}{2} \times \frac{1}{1-\frac{x}{2}} - \frac{1}{1+x} = -\frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^2 - \sum_{n=0}^{\infty}(-x)^n$$
$$= \sum_{n=0}^{\infty}(-\frac{1}{2^{n+1}} - (-1)^n)x^n$$

   The series converges when $|\frac{x}{2}| < 1$ and $|x| < 1$, so the interval of convergence is $(-1, 1)$.

   (iv). First observe that

$$\frac{1}{(1+4x)^2} = (-\frac{1}{4}\frac{1}{1+4x})' = (-\frac{1}{4}\sum_{n=0}^{\infty}(-4x)^n)' = (\sum_{n=0}^{\infty}(-4)^{n-1}x^n)' = \sum_{n=1}^{\infty}(-4)^{n-1}nx^{n-1}$$

So

$$\frac{x}{(1+4x)^2} = x\sum_{n=1}^{\infty}(-4)^{n-1}nx^{n-1} = \sum_{n=1}^{\infty}(-4)^{n-1}nx^n$$

The series converges when $|4x| < 1$, so the interval of convergence is $(-\frac{1}{4}, \frac{1}{4})$.

2. Find the Taylor series of $f(x) = \ln x$ at 1, and prove $f(x)$ equals to this Taylor series on $(\frac{1}{2}, \frac{3}{2})$.

   Solution: $f^{(x)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$ for all $n \geq 1$.
   $f^{(n)}(1) = (-1)^{n-1}(n-1)!$, so the Taylor series is

$$T(x) = f(1) + \sum_{n=1}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n$$

$$|R_N| = |\frac{f^{(N+1)}(z)}{(N+1)!}(x-1)^{N+1}| = \frac{1}{N+1}|\frac{x-1}{z}|^{N+1}$$

   for some z between x and 1. If $\frac{1}{2} < x < \frac{3}{2}$, then $|x-1| < \frac{1}{2}$ and $z > \frac{1}{2}$, so $|\frac{x-1}{z}| < 1$, $\lim\limits_{n\to\infty}|R_N| = 0$, we conclude $T(x)$ equals to $f(x)$ on $(\frac{1}{2}, \frac{3}{2})$.

3. Use binomial series to expand the function $f(x) = \frac{1}{(2+x)^3}$ as a power series, and state the radius of convergence.

   Solution:

$$\begin{aligned}
f(x) = \frac{1}{(2+x)^3} &= \frac{1}{8}(1+\frac{x}{2})^{-3} = \frac{1}{8}(1+\sum_{n=1}^{\infty}\binom{-3}{n}(\frac{x}{2})^n)) \\
&= \frac{1}{8} + \sum_{n=1}^{\infty}\frac{(-3)\times(-4)\times ...\times(-3-n+1)}{n!}\frac{x^n}{2^{n+3}} \\
&= \frac{1}{8} + \sum_{n=1}^{\infty}\frac{(-1)^n 3\times 4\times ...\times(n+2)}{(n)!}\frac{x^n}{2^{n+3}} \\
&= \frac{1}{8} + \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2}\frac{x^n}{2^{n+3}} \\
&= \frac{1}{8} + \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n
\end{aligned}$$

4. Use Taylor series to evaluate the limit

$$\lim_{n\to\infty} \frac{1-\cos x}{1+x-e^x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} &= \lim_{x\to 0} \frac{1-(1-\frac{x^2}{2}+\frac{x^4}{24}-...)}{1+x-(1+x+\frac{x^2}{2}+\frac{x^3}{6}+...)} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2}-\frac{x^4}{24}+...}{-\frac{x^2}{2}-\frac{x^3}{6}-...} \\
&= \lim_{x\to 0} \frac{\frac{1}{2}-\frac{x^2}{24}+...}{-\frac{1}{2}-\frac{x}{6}-...} \\
&= \frac{\frac{1}{2}+0}{-\frac{1}{2}+0} \\
&= -1
\end{aligned}$$

5. Evaluate the integral as as infinite series:

$$\int x\cos(x^3)\,dx$$

Solution:

$$\begin{aligned}
\int x\cos(x^3)\,dx &= \int x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^3)^{2n}\,dx \\
&= \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n+1}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{6n+1}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{6n+2}}{6n+2} + C
\end{aligned}$$

6. Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{4^n}{n5^n}$$

Solution:
We know $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$ for $|x| < 1$, so take $x = \frac{4}{5}$:
$$-\ln 5 = \ln(\frac{1}{5}) = \ln(1 - \frac{4}{5}) = -\sum_{n=1}^{\infty} \frac{1}{n}(\frac{4}{5})^n$$

We get
$$\ln 5 = \sum_{n=1}^{\infty} \frac{1}{n}(\frac{4}{5})^n$$