1. Determine whether the following series is convergent or divergent.

   (i). $\sum \frac{n!}{2^{n^2}}$

   (ii). $\sum (\frac{n^2+1}{2n^2+1})^n$

   Solution:

   (i).

$$\lim_{n\to\infty} |\frac{\frac{(n+1)!}{2^{(n+1)^2}}}{\frac{n!}{2^{n^2}}}| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \times \frac{2^{n^2}}{2^{n^2+2n+1}} = \lim_{n\to\infty} \frac{n+1}{2^{2n+1}}$$

   By the L'Hospital's Rule,

$$\lim_{x\to+\infty} \frac{x+1}{2^{2x+1}} = \lim_{x\to+\infty} \frac{1}{2 \times 2^{2x+1} \ln 2} = 0$$

   So

$$\lim_{n\to\infty} |\frac{\frac{(n+1)!}{2^{(n+1)^2}}}{\frac{n!}{2^{n^2}}}| = \lim_{n\to\infty} \frac{n+1}{2^{2n+1}} = 0 < 1$$

   By the Ratio Test, we conclude the series converges.

   (ii).

$$\lim_{\lim\to\infty} \sqrt[n]{|(\frac{n^2+1}{2n^2+1})^n|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$$

   By the Root Test, we conclude the series converges.

2. Determine whether the following series is absolutely convergent, conditionally convergent or divergent.

   (i). $\sum \frac{(-1)^n \tan^{-1} n}{n^2}$

   (ii). $\sum (\frac{-2}{n})^n$

   Solution:

(i). Note that $|\tan^{-1} x| < \frac{\pi}{2}$, so $0 \le |\frac{(-1)^n \tan^{-1} n}{n^2}| \le \frac{\pi}{2}\frac{1}{x^2}$. By Comparison Test, the convergence of $\sum \frac{\pi}{2}\frac{1}{x^2}$ implies the convergence of $\sum |\frac{(-1)^n \tan^{-1} n}{n^2}|$, so $\sum \frac{(-1)^n \tan^{-1} n}{n^2}$ converges absolutely.

(ii).

$$\lim_{n\to\infty} \sqrt[n]{|(\frac{-2}{n})^n|} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

So by the Ratio Test, the series $\sum (\frac{-2}{n})^n$ converges absolutely.

3. $\{b_n\}$ is a sequence and $\lim_{n\to\infty} b_n = \frac{1}{2}$. Determine whether the given series is absolutely convergent, conditionally convergent or divergent.

$$\sum \frac{(-1)^n n!}{n^n b_1 b_2 ... b_n}$$

Solution:

$$\lim_{n\to\infty} |\frac{\frac{(-1)^{n+1} n+1!}{n+1^{n+1} b_1 b_2 ... b_n b_{n+1}}}{\frac{(-1)^n n!}{n^n b_1 b_2 ... b_n}}| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{n^n b_1 ... b_n}{(n+1)^{n+1} b_1 ... b_n b_{n+1}} = \lim_{n\to\infty} (\frac{n}{n+1})^n \frac{1}{b_{n+1}} = \frac{2}{e} < 1$$

By the Ratio Test, the series converges absolutely.

4. Find all the values for k such that the series

$$\sum \frac{(n!)^2}{(kn)!}$$

converges.

Solution:

When $k = 1$: the series becomes $\sum \frac{(n!)^2}{(n)!} = n!$, and $n! \neq 0$, so the series diverges.

When $k \geq 2$:

$$\lim_{n\to\infty} |\frac{\frac{((n+1)!)^2}{(k(n+1)!)}}{\frac{(n!)^2}{(kn)!}}| = \lim_{n\to\infty} (\frac{(n+1)!}{n!})^2 \frac{(kn)!}{(kn+k)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(kn+1)(kn+2)...(kn+k)}$$
$$\leq \lim_{n\to\infty} \frac{(n+1)^2}{(kn+1)(kn+2)}$$
$$\leq \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$$

By the Ratio Test, it converges.

We conclude the series converges for $k \geq 2$.

5. Find the radius of convergence and interval of convergence of the power series.

(i). $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

(ii). $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Solution:

(i). The radius of convergence is

$$R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{\frac{n}{4^n}}{\frac{(n+1)}{4^{n+1}}} = \lim_{n\to\infty} \frac{n}{n+1} \times 4 = 4$$

When $x = -1 + 4 = 3$, the series is $\displaystyle\sum_{n=0}^{\infty} n$, which diverges since $\displaystyle\lim_{n\to\infty} n \neq 0$

When $x = -1 - 4 = -5$, the series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n n$, which is divergent since $\displaystyle\lim_{n\to\infty} (-1)^n n \neq 0$.

So the interval of convergence is $(-5, 3)$.

(ii).

$$\lim_{n\to\infty} |\frac{(-1)^{n+1} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{(-1)^n \frac{x^{2n+1}}{(2n+1)!}}| = \lim_{n\to\infty} \frac{x^{2n+3}}{(2n+3)!} \times \frac{(2n+1)!}{x^{2n+1}} = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 < 1$$

for any real number x, so the radius of convergence is ∞, and the interval of convergence is $(-\infty, +\infty)$.

6. Let p and q be real numbers with $p < q$. Find a power series whose interval of convergence is $[p, q)$.

   Solution: Consider the power series

$$\sum_{n=0}^{\infty} (\frac{2}{q-p})^n \frac{1}{n+1} (x - \frac{p+q}{2})^n$$

   The radius of convergence is

$$R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{(\frac{2}{q-p})^n \frac{1}{n+1}}{(\frac{2}{q-p})^{n+1} \frac{1}{n+2}} = \lim_{n\to\infty} (\frac{q-p}{2}) \frac{n+2}{n+1} = \frac{q-p}{2}$$

   When $x = \frac{p+q}{2} + \frac{q-p}{2} = q$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which is divergent.

   When $x = \frac{p+q}{2} - \frac{q-p}{2} = p$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, which is convergent by Alternating Convergence Test.

   So the interval of convergence is $[p, q)$