1. Determine if the following sequences converge or diverge:

(i). $\{\tan \frac{3}{n}\}$

(ii). $\{\frac{(\ln n)^2}{n}\}$

(iii). $\{(1+\frac{2}{n})^n\}$

(iv). $\{\frac{\sin 3n}{2^n-1}\}$

Solution:

(i). $\lim\limits_{n\to\infty} \dfrac{3}{n} = 0$ and $f(x) = \tan x$ is continuous at $x = 0$, so

$$\lim_{n\to\infty} \tan(\frac{3}{n}) = \tan(\lim_{n\to\infty} \frac{3}{n}) = \tan 0 = 0$$

So the sequence converges.

(ii). By the L'Hospital's Rule,

$$\lim_{x\to+\infty} \frac{(\ln x)^2}{x} = \lim_{x\to+\infty} \frac{2\ln x}{x} = \lim_{x\to+\infty} \frac{2}{x} = 0$$

So $\lim\limits_{n\to\infty} \dfrac{(\ln n)^2}{n} = \lim\limits_{x\to+\infty} \dfrac{(\ln x)^2}{x} = 0$, the sequence converges.

(iii).

$$\lim_{x\to+\infty} (1+\frac{2}{x})^x = \lim_{y\to+\infty} (1+\frac{2}{2y})^{2y} = \lim_{y\to+\infty} ((1+\frac{1}{y})^y)^2 = (\lim_{y\to+\infty} (1+\frac{1}{y})^y)^2 = e^2$$

So $\lim\limits_{n\to\infty} (1+\dfrac{2}{n})^n = \lim\limits_{x\to+\infty} (1+\dfrac{2}{x})^x = e^2$, the sequence converges.

(iv). $-\frac{1}{2^n-1} \leq \frac{\sin 3n}{2^n-1} \leq \frac{1}{2^n-1}$, and

$$\lim_{n\to\infty} -\frac{1}{2^n-1} = \lim_{n\to\infty} \frac{1}{2^n-1} = 0$$

By Squeeze Theorem, $\lim\limits_{n\to\infty} \dfrac{\sin 3n}{2^n-1} = 0$, so the sequence converges.

2. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$. Show that the sequence converges.

   Solution: We first show $a_n \leq 2$ for all n by induction:

   First, $a_1 = \sqrt{2} < 2$.

   Next, if $a_n < \sqrt{2}$, then $a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2+2} = 2$.

   So $\{a_n\}$ is bounded above by 2.

   $\{a_n\}$ is increasing since $\frac{a_{n+1}}{a_n} = \frac{\sqrt{2+a_n}}{a_n} = \sqrt{\frac{2}{a_n^2} + \frac{1}{a_n}} > \sqrt{2 \times (\frac{1}{2})^2 + \frac{1}{2}} = 1$.

   So by the Monotonic Convergence Theorem, conclude $\{a_n\}$ converges.

3. Show that if $\{a_n\}$ is bounded and $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty} a_n b_n = 0$.

   Solution: If $\{a_n\}$ is bounded, then there exists constant $K > 0$ such that $|a_n| < K$ for all n. This implies $-K|b_n| \leq a_n b_n \leq K|b_n|$, and $\lim\limits_{n\to\infty} -K|b_n| = \lim\limits_{n\to\infty} K|b_n| = 0$, so by Squeeze Theorem, $\lim\limits_{n\to\infty} a_n b_n = 0$.

4. Determine if the following series converge or diverge:

   (i). $\sum (-1)^n n$

   (ii). $\sum \ln(1 + \frac{1}{n})$

   (iii). $\sum \frac{1}{2n-1}$

   (iv). $\sum \frac{2+(-1)^n}{n^2}$

   (v). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

   (vi). $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

   Solution:

   (i). $\lim (-1)^n n \neq 0$, so the series diverges.

   (ii). $\displaystyle S_N = \sum_{n=1}^{N} \ln(1 + \frac{1}{n}) = \sum_{n=1}^{N} \ln(\frac{n+1}{n}) = \ln(\frac{2}{1} \times \frac{3}{2} \times ... \times \frac{N+1}{N}) = \ln(N+1)$

   So $\lim\limits_{N\to\infty} S_N = +\infty$, the series diverges.

(iii). $\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0$, and $\sum \frac{1}{n}$ diverges, so by Limiting Convergence Test, $\sum \frac{1}{2n-1}$ diverges.

(iv). $0 < \frac{2+(-1)^n}{n^2} \leq \frac{3}{n^2}$, and $\sum \frac{3}{n^2}$ converges, so by the Comparison Test, $\sum \frac{2+(-1)^n}{n^2}$ converges.

(v). $f(x) = \frac{1}{x\ln x}$ is a positive decreasing function on $[2, +\infty)$.

$$\int_2^{+\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to+\infty} \int_2^t \frac{1}{x\ln x}\,dx = \lim_{t\to+\infty} \int_2^t \frac{1}{\ln x}\,d\ln x = \lim_{t\to+\infty} \ln\ln x\Big|_2^t = +\infty$$

So the series diverges.

(vi). $\frac{1}{\ln n}$ is a positive decreasing sequence for $n \geq 2$, so by the Alternating Convergence Test, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges.