1. Solve the differential equation:

$$y' = \frac{\ln x}{xy}$$

with initial condition $x = 1, y = 2$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\ln x}{xy} \\
y\,dy &= \frac{\ln x}{x}\,dx \\
\int y\,dy &= \int \frac{\ln x}{x}\,dx \\
\frac{1}{2}y^2 &= \frac{1}{2}(\ln x)^2 + C
\end{aligned}$$

Plug in $x = 1, y = 2$, we see $\frac{1}{2} \times 2^2 = \frac{1}{2}(\ln 1)^2 + C$, so $C = 2$, and $y^2 = (\ln x)^2 + 4$. Since when $x = 1$, $y = 2 > 0$, when taking square root, we should take the positive one, we thus conclude

$$y = \sqrt{(\ln x)^2 + 4}$$

2. Solve the differential equation:

$$y' - 2y = e^x$$

Solution:

$$\begin{aligned}
y' - 2y &= e^x \\
e^{-2x} - 2e^{-2x}y &= e^{-2x}e^x \\
(e^{-2x}y)' &= e^{-x} \\
e^{-2x}y &= -e^{-x} + C \\
y &= -e^x + Ce^{2x}
\end{aligned}$$

So the solution is

$$y = -e^x + Ce^{2x}$$

where C is a constant.

3. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

   Write a differential equation that is satisfied by y, and solve it.

   Solution: The differential equation is

   $$y' = ky(1-y)$$

   where k is some positive constant.

   $$\begin{aligned} \frac{dy}{dx} &= ky(1-y) \\ \frac{1}{y(1-y)}\,dy &= k\,dx \\ \int \frac{1}{y(1-y)}\,dy &= \int k\,dx \\ \ln(y(1-y)) &= kt + C \\ y(1-y) = e^{kt+C} \end{aligned}$$

   Solving the last equation, we obtain

   $$y = \frac{1 \pm \sqrt{1-4e^{kt+C}}}{2}$$

   Since y should be an increasing function with respect to t, we conclude

   $$y = \frac{1 - \sqrt{1-4e^{kt+C}}}{2}$$

4. Determine whether the following sequences converge or diverge. If it converges, find the limit:

   (i). $\{\frac{n^3}{n^3+1}\}$

   (ii). $\{n^2 e^{-n}\}$

   (iii). $\{\sin \frac{n\pi}{2}\}$

   (iv). $\{\frac{(-3)^n}{n!}\}$

Solution:

(i). $\lim_{n\to\infty} \dfrac{n^3}{n^3+1} = \lim_{n\to\infty} \dfrac{1}{1+\frac{1}{n^3}} = \dfrac{1}{1} = 1$

(ii). $\lim_{x\to+\infty} x^2e^{-x} = \lim_{x\to+\infty} \dfrac{x^2}{e^x} = \lim_{x\to+\infty} \dfrac{2x}{e^x} = \lim_{x\to+\infty} \dfrac{2}{e^x} = 0$, so

$$\lim_{n\to\infty} n^2e^{-n} = 0$$

(iii). We see

$$\sin\frac{n\pi}{2} = \begin{cases} 1, & \text{for } n = 4k+1 \\ 0, & \text{for } n = 2k \\ -1, & \text{for } n = 4k+3 \end{cases}$$

So the sequence diverges.

(iv). For $n \geq 4$, $0 < \frac{3^n}{n!} = \frac{3^3}{3!} \times \frac{3^{n-3}}{4\times\ldots\times n} \leq \frac{9}{2}(\frac{3}{4})^{n-3}$

$\lim_{n\to\infty} \dfrac{9}{2}(\dfrac{3}{4})^{n-3} = 0$, so by Squeeze Theorem, $\lim_{n\to\infty} \dfrac{3^n}{n!} = 0$, and $|\frac{(-3)^n}{n!}| = \frac{3^n}{n!}$, so we conclude

$$\lim_{n\to\infty} \frac{(-3)^n}{n!} = 0$$

5. Find the limit of the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

Solution: Let $L = \lim_{n\to\infty} a_n$. The sequence satisfies the relation

$$a_{n+1} = \sqrt{2a_n}$$

So

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n}$$

$$L = \sqrt{2L}$$

We get $L = 2$ or $L = 0$, but $L = 0$ is impossible since $a_n \geq \sqrt{2}$ for all n. So we conclude

$$\lim_{n\to\infty} a_n = 2$$