1. Find the length of the curve $y^2 = 4(x+4)^3$, $0 \le x \le 2$, $y > 0$.

   Solution: If $y > 0$, $y = 2(x+4)^{\frac{3}{2}}$, so $y' = 3(x+4)^{\frac{1}{2}}$, the arc length is

$$\begin{aligned}\int_0^2 \sqrt{1+(y')^2}\,dx &= \int_0^2 \sqrt{1+9(x+4)}\,dx \\ &= \int_0^2 \sqrt{9x+37}\,dx \\ &= \frac{1}{9}\int_0^2 \sqrt{9x+37}\,d(9x+37) \\ &= \frac{2}{27}(55^{\frac{3}{2}} - 37^{\frac{3}{2}})\end{aligned}$$

2. Find the arc length function of the curve $y = \sin^{-1} x + \sqrt{1-x^2}$ with starting point at $(0,1)$

   Solution: $y' = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} = \sqrt{\frac{1-x}{1+x}}$.

   So the arc length function is

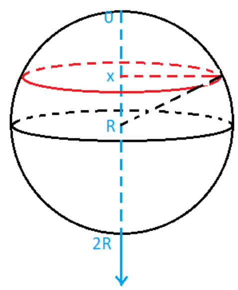
$$\begin{aligned}s(x) = \int_0^x \sqrt{1+y'(t)^2}\,dt &= \int_0^x \sqrt{1+\frac{1-t}{1+t}}\,dt \\ &= \int_0^x \frac{\sqrt{2}}{\sqrt{1+t}}\,d(1+t) \\ &= 2\sqrt{2}\sqrt{1+t}\Big|_0^x \\ &= 2\sqrt{2}(\sqrt{1+x}-1)\end{aligned}$$

3. A spherical tank of radius R meters is full of water. If the density of water is ρ and gravitational acceleration is g, Find the work required to pump the water out from the top of the sphere.

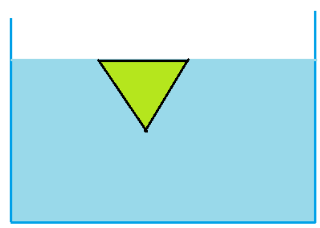
   Solution:

   We take a downward x-axis with origin at the top of the sphere. The horizontal section at x is a disk of radius $\sqrt{R^2-(R-x)^2} = \sqrt{2Rx-x^2}$, so the total work needed is

$$\int_0^{2R} \rho g \pi (2Rx - x^2) x \, dx = \pi \rho g \int_0^{2R} 2Rx^2 - x^3 \, dx$$
$$= \pi \rho g \left(\frac{16}{3} R x^3 - \frac{1}{4} x^4 \Big|_0^{2R} \right)$$
$$= \frac{4}{3} \pi \rho g R^4$$



4. A plate is in the shape of an equilateral triangle with length of edge d. It is submerged into water vertically with one edge at the surface of water. If the density of water is ρ and gravitational acceleration is g, compute the hydrostatic force against one side of the plate.



Solution: We choose x-axis downward with origin at the surface of water, then the horizontal section at depth x is $2 \times (\frac{\sqrt{3}}{2} d - x) \times \tan \frac{\pi}{6} = d - \frac{2}{\sqrt{3}} x$.

So the hydrostatic force is

$$\int_0^{\frac{\sqrt{3}}{2} d} \rho g x (d - \frac{2}{\sqrt{3}} x) \, dx = \frac{1}{8} \rho g d^3$$

5. Find the centre of mass of the region bounded by $y = x^2$, x-axis and $x = 1$.

   Solution:

   $\int_0^1 x(x)^2\,dx = \frac{1}{4}$, $\int_0^1 \frac{(x^2)^2}{2}\,dx = \frac{1}{10}$, $\int_0^1 x^2\,dx = \frac{1}{3}$. So the centre of mass is

$$(\frac{\frac{1}{4}}{\frac{1}{3}}, \frac{\frac{1}{10}}{\frac{1}{3}}) = (\frac{3}{4}, \frac{3}{10})$$

6. Find the centre of mass of the region bounded between the circles $(x+1)^2+y^2 = 1$ and $x^2 + y^2 = 4$.

   Solution:

   By the Symmetric Principle, the centre of mass of the small circle is at $(-1, 0)$, and the centre of mass of the big circle is at $(0, 0)$. The small circle has area π and the big circle has area 4π, so the area between them is 3π. Assume the centre of mass of the area between the circles is (x, y), then

$$\begin{cases} \pi \times (-1) + 3\pi x = 4\pi \times 0 \\ \pi \times 0 + 3\pi x = 4\pi \times 0 \end{cases}$$

   We get $x = \frac{1}{3}$, $y = 0$. So the centre of mass of the region between the circles is $(\frac{1}{3}, 0)$.