1. f is a function defined on $[0,1]$. Some of its values are shown in the table below:

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	2	2.5	3.5	3	4	4.5	3.5	2	1.5

Approximate $\int_0^1 f(x)\,dx$ by the following rules:

(i). Midpoint Rule for $n = 4$

$$\begin{aligned}\int_0^1 f(x)\,dx &\approx \frac{1}{4}(f(0.125) + f(0.375) + f(0.625) + f(0.875)) \\ &= \frac{1}{4}(2.5 + 3 + 4.5 + 2) \\ &= 3\end{aligned}$$

(ii). Trapezoid Rule for $n = 4$

$$\begin{aligned}\int_0^1 f(x)\,dx &\approx \frac{1}{2\times 4}(f(0) + 2f(0.25) + f(0.5) + 2f(0.75) + f(1)) \\ &= \frac{1}{8}(2 + 2\times 3.5 + 2\times 4 + 2\times 3.5 + 1.5) \\ &= 3.1875\end{aligned}$$

(iii). Simpson's Rule for $n = 8$

Solution:

$$\begin{aligned}\int_0^1 f(x)\,dx &\approx \frac{1}{3\times 8}(f(0) + 4f(0.125) + 2f(0.25) + 4f(0.375) \\ &+ 2f(0.5) + 4f(0.625) + 2f(0.75) + 4f(0.875) + f(1)) \\ &= \frac{1}{24}(2 + 4\times 2.5 + 2\times 3.5 + 4\times 3 \\ &+ 2\times 4 + 4\times 4.5 + 2\times 3.5 + 4\times 2 + 1.5) \\ &= 3.0625\end{aligned}$$

2. How large should n be to guarantee that the Midpoint Rule Approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within $0.00001 = 10^{-5}$?

Solution: The error formula for the Midpoint Rule is $|E_T| \le \frac{K(b-a)^3}{24n^2}$, where $|f''(x)| \le K$ on $[a,b]$.

$f(x) = e^{x^2}$, $f'(x) = 2xe^{x^2}$, $f''(x) = 2(1+2x^2)e^{x^2}$. Since $f''(x)$ is an increasing function, on $[0,1]$ we have $|f''(x)| = f''(x) \le f''(1) = 6e$, so we can take $K = 6e$.

$$\frac{6e \times (1-0)^3}{24n^2} \le 10^{-5}$$

We get $n \ge 50\sqrt{10e} \approx 260.7$, so we need to take $n = 261$.

3. Determine if each of the following improper integrals is convergent or divergent, and evaluate if it is convergent:

(i). $\int_3^{+\infty} \frac{1}{(x-2)^{\frac{3}{2}}}\,dx$

Solution:

$$\int_3^t \frac{1}{(x-2)^{\frac{3}{2}}}\,dx = \int_3^t \frac{1}{(x-2)^{\frac{3}{2}}}\,d(x-2) = -\frac{2}{\sqrt{x-2}}\bigg|_3^t = 2 - \frac{2}{\sqrt{t-2}}$$

$$\int_3^{+\infty} \frac{1}{(x-2)^{\frac{3}{2}}}\,dx = \lim_{t\to+\infty}\int_3^t \frac{1}{(x-2)^{\frac{3}{2}}}\,dx = \lim_{t\to+\infty} 2 - \frac{2}{\sqrt{t-2}} = 2$$

(ii). $\int_0^{+\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx$

Solution:

$$\int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx = \frac{1}{3}\int_0^t \frac{1}{\sqrt{1+x^3}}\,d(1+x^3) = \frac{2}{3}(\sqrt{1+t^3}-1)$$

So

$$\int_0^{+\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to+\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to+\infty}\frac{2}{3}(\sqrt{1+t^3}-1) = +\infty$$

The improper integral diverges.

(iii). $\int_{-\infty}^{+\infty} xe^{-x^2}\,dx$

Solution:

$$\int_0^t xe^{-x^2}\,dx = -\frac{1}{2}\int_0^t e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}\Big|_0^t = \frac{1-e^{-t^2}}{2}$$

$$\int_0^{+\infty} xe^{-x^2}\,dx = \lim_{t\to+\infty}\int_0^t xe^{-x^2}\,dx = \lim_{t\to+\infty}\frac{1-e^{-t^2}}{2} = \frac{1}{2}$$

$$\int_t^0 xe^{-x^2}\,dx = -\frac{1}{2}\int_t^0 e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}\Big|_t^0 = \frac{e^{-t^2}-1}{2}$$

$$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\int_t^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\frac{e^{-t^2}-1}{2} = -\frac{1}{2}$$

So

$$\int_{-\infty}^{+\infty} xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^{+\infty} xe^{-x^2}\,dx = \frac{1}{2}-\frac{1}{2} = 0$$

(iv). $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Solution:

$$\begin{aligned}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx &= 2\int_t^1 \ln x d\sqrt{x} \\ &= 2(\sqrt{x}\ln x\Big|_t^1 - \int_t^1 \sqrt{x}\,d\ln x) \\ &= 2(-\sqrt{t}\ln t - 2\sqrt{x}\Big|_t^1) \\ &= 2(-\sqrt{t}\ln t - 2 + 2\sqrt{t})\end{aligned}$$

By the L'Hospital's Rule,

$$\lim_{t\to0^+}\sqrt{t}\ln t = \lim_{t\to0^+}\frac{\ln t}{t^{-\frac{1}{2}}} = \lim_{t\to0^+}\frac{t^{-1}}{(-\frac{1}{2})t^{-\frac{3}{2}}} = \lim_{t\to0^+} -2\sqrt{t} = 0$$

So

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+} 2(-\sqrt{t}\ln t - 2 + 2\sqrt{t}) = -4$$

(v). $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution:

$$\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\sin^{-1} t} \frac{1}{\sqrt{1-\sin^2\theta}}\,d\sin\theta = \int_0^{\sin^{-1} t} 1\,d\theta = \sin^{-1} t$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1^-}\sin^{-1} t = \frac{\pi}{2}$$

(vi). $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

Solution:

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^t \frac{1}{\sqrt[3]{x-1}}\,d(x-1) = \frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_0^t = \frac{3}{2}((t-1)^{\frac{2}{3}}-1)$$

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-}\frac{3}{2}((t-1)^{\frac{2}{3}}-1) = -\frac{3}{2}$$

$$\int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_t^9 \frac{1}{\sqrt[3]{x-1}}\,d(x-1) = \frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_t^9 = \frac{3}{2}(4-(t-1)^{\frac{2}{3}})$$

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to 1^-}\frac{3}{2}(4-(t-1)^{\frac{2}{3}}) = 6$$

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2}+6 = \frac{9}{2}$$

(vii). $\int_{-1}^1 \frac{e^x}{e^x-1}\,dx$

Solution:

$$\int_{-1}^1 \frac{e^x}{e^x-1}\,dx = \int_{-1}^0 \frac{e^x}{e^x-1}\,dx + \int_0^1 \frac{e^x}{e^x-1}\,dx$$

$$\int_{-1}^{t} \frac{e^x}{e^x-1}\,dx = \int_{-1}^{t} \frac{1}{e^x-1}\,d(e^x-1) = \ln|e^x-1|\Big|_{-1}^{t} = \ln|e^t-1| - \ln|e^{-1}-1|$$

So

$$\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-}\int_{-1}^{t} \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-} \ln|e^t-1| - \ln|e^{-1}-1| = -\infty$$

diverges, which implies $\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx$ diverges.

(viii). $\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)}\,dx$

Solution:

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^{1} \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^{+\infty} \frac{1}{\sqrt{x}(1+x)}\,dx$$

$$\int_t^{1} \frac{1}{\sqrt{x}(1+x)}\,dx = \int_{\sqrt{t}}^{1} \frac{1}{u(1+u^2)}\,du^2 = 2\int_{\sqrt{t}}^{1} \frac{1}{1+u^2}\,du = 2(\tan^{-1}u)\Big|_{\sqrt{t}}^{1} = 2(\frac{\pi}{4}-\tan^{-1}\sqrt{t})$$

$$\int_0^{1} \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to0^+} 2(\frac{\pi}{4}-\tan^{-1}\sqrt{t}) = \frac{\pi}{2}$$

$$\int_1^{t} \frac{1}{\sqrt{x}(1+x)}\,dx = \int_1^{\sqrt{t}} \frac{1}{u(1+u^2)}\,du^2 = 2\int_1^{\sqrt{t}} \frac{1}{1+u^2}\,du = 2(\tan^{-1}u)\Big|_{1}^{\sqrt{t}} = 2(\tan^{-1}\sqrt{t}-\frac{\pi}{4})$$

$$\int_1^{+\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{t\to+\infty} 2(\tan^{-1}\sqrt{t}-\frac{\pi}{4}) = \frac{\pi}{2}$$

So

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^{1} \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^{+\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

4. Show that
$$\int_0^{+\infty} x^2 e^{-x^2}\,dx = \frac{1}{2}\int_0^{+\infty} e^{-x^2}\,dx$$

Solution:

$$\begin{aligned}
\int_0^t x^2 e^{-x^2\,dx} &= -\frac{1}{2}\int x(-2x)e^{-x^2}\,dx \\
&= -\frac{1}{2}\int x\,de^{-x^2} \\
&= -\frac{x}{2}e^{-x^2}\Big|_0^t + \frac{1}{2}\int_0^t e^{-x^2}\,dx \\
&= -\frac{t}{2e^{t^2}} + \frac{1}{2}\int_0^t e^{-x^2}\,dx
\end{aligned}$$

Taking the limit $t \to +\infty$, we get

$$\lim_{t\to+\infty}\int_0^t x^2 e^{-x^2\,dx} = -\lim_{t\to+\infty}\frac{t}{2e^{t^2}} + \lim_{t\to+\infty}\frac{1}{2}\int_0^t e^{-x^2}\,dx$$

By the L'Hospital's Rule, $\lim_{t\to+\infty}\frac{t}{2e^{t^2}} = \lim_{t\to+\infty}\frac{1}{4te^{t^2}} = 0$, so we conclude

$$\int_0^{+\infty} x^2 e^{-x^2\,dx} = \frac{1}{2}\int_0^{+\infty} e^{-x^2}\,dx$$