1. Compute $\int_1^4 \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx$

   Solution:

   Let $x = t^2$, then

$$\begin{aligned}\int_1^4 \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx = \int_1^2 \frac{t^2-t}{t^2+t}\,dt^2 &= \int_1^2 \frac{t(t-1)}{t(t+1)} \times 2t\,dt \\ &= \int_1^2 \frac{t-1}{t+1} \times 2t\,dt \\ &= \int_1^2 \frac{(t+1)-2}{t+1} \times 2t\,dt \\ &= \int_1^2 (1-\frac{2}{t+1}) \times 2t\,dt \\ &= \int_1^2 2t - \frac{4t}{t+1}\,dt \\ &= \int_1^2 2t - 4 + \frac{4}{t+1}\,dt \\ &= t^2 - 4t + 4\ln(t+1)\Big|_1^2 \\ &= -1 + 4\ln 3 - 4\ln 2\end{aligned}$$

2. Compute $\int \sqrt{1+\sqrt{x}}\,dx$

   Solution: Let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\,dx$

$$\begin{aligned}\int \sqrt{1+\sqrt{x}}\,dx &= \int \sqrt{u}(2\sqrt{x})\,du = 2\int u^{\frac{1}{2}}(u-1)\,du \\ &= 2\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \\ &= \frac{4}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + C \\ &= \frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C\end{aligned}$$

3. Compute $\int x^2 \ln x \, dx$

Solution:

$$\begin{aligned}\int x^2 \ln x \, dx &= (x^2 \ln x)x - \int x \, d(x^2 \ln x) \\ &= x^3 \ln x - \int x(2x \ln x + x) \, dx \\ &= x^3 \ln x - 2\int x^2 \ln x \, dx - \int x^2 \, dx \\ &= x^3 \ln x - 2\int x^2 \ln x - \frac{1}{3}x^3\end{aligned}$$

4. Compute $\int \frac{\ln x}{x^2} \, dx$

Solution:

$$\begin{aligned}\int \frac{\ln x}{x^2} \, dx = -\int \ln x \, d\frac{1}{x} = -\frac{\ln x}{x} + \int \frac{1}{x} \, d\ln x &= -\frac{\ln x}{x} + \int \frac{1}{x^2} \, dx \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C \\ &= -\frac{\ln x + 1}{x} + C\end{aligned}$$

5. Compute $\int x^8 \sqrt{x^3 - 1} \, dx$

Solution:

Let $u = x^3 - 1$, then $du = (x^3 - 1)' dx = 3x^2 dx$

$$\begin{aligned}\int x^8 \sqrt{x^3 - 1} \, dx &= \frac{1}{3} \int (x^3)^2 \sqrt{x^3 - 1}(3x^2) \, dx \\ &= \frac{1}{3} \int (u + 1)^2 u^{\frac{1}{2}} \, du \\ &= \frac{1}{3} \int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \, du \\ &= \frac{1}{3}(\frac{2}{7}u^{\frac{7}{2}} + 2 \times \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}) + C \\ &= \frac{2}{21}(x^3 - 1)^{\frac{7}{2}} + \frac{4}{15}(x^3 - 1)^{\frac{5}{2}} + \frac{2}{9}(x^3 - 1)^{\frac{3}{2}} + C\end{aligned}$$

6. Compute $\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

   Solution: Let $u=\sqrt{x}$, then $x=u^2,\ dx=2u\,du$

$$\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_1^2 \frac{\sin u}{u} 2u\,du = \int_1^2 2\sin u\,du = 2(\cos 1-\cos 2)$$

7. Compute $\int_6^8 x\sqrt{100-x^2}\,dx$

   Solution: Let $u=100-x^2$, $du=-2x\,dx$

$$\begin{aligned}\int_6^8 x\sqrt{100-x^2}\,dx &= \int_{64}^{36} -\frac{1}{2}\sqrt{u}\,du\\ &= \frac{1}{2}\int_{36}^{64}\sqrt{u}\,du\\ &= \frac{1}{3}u^{\frac{3}{2}}\Big|_{36}^{64}\\ &= \frac{296}{3}\end{aligned}$$

8. Compute $\int \tan^{-1}2x\,dx$

   Solution:

$$\begin{aligned}\int \tan^{-1}2x\,dx = x\tan^{-1}2x - \int x(\tan^{-1}2x)'\,dx &= x\tan^{-1}2x - \int x(\frac{2}{1+4x^2})\,dx\\ &= x\tan^{-1}2x - \frac{1}{4}\int \frac{1}{1+4x^2}\,d(1+4x^2)\\ &= x\tan^{-1}2x - \frac{1}{4}\ln(1+4x^2)+C\end{aligned}$$

9. f is a continuously differentiable function. Show that

$$\int_a^b f'(x)e^{f(x)}\,dx = e^{f(b)}-e^{f(a)}$$

   Solution: Let $u=f(x),\ du=f'(x)\,dx$

$$\int_a^b f'(x)e^{f(x)}\,dx = \int_{f(a)}^{f(b)} e^u\,du = e^u\Big|_{f(a)}^{f(b)} = e^{f(b)}-e^{f(a)}$$

10. Find the area bounded by the function $f(x) = \frac{\ln x}{x}$ and x-axis over the interval $[1, e]$

    Solution:

$$\int_1^e \frac{\ln x}{x}\, dx = \int_1^e \ln x \, d \ln x = \int_{\ln 1}^{\ln e} u\, du = \int_0^1 u\, du = \frac{1}{2}$$