Series

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1 Series

Given a sequence $\{a_n\}$, we consider the sequence $\{s_N\}$ where $s_N = \sum_{n=1}^{N} a_n = a_1 + ... + a_N$, and we write $\sum_{n=1}^{\infty} a_n = \lim\limits_{N\to\infty} s_N = \lim\limits_{N\to\infty} \sum\limits_{n=1}^{N} a_n$. $\sum_{n=1}^{\infty} a_n$ is called a **series**, and when the limit exists, we say the series converges, otherwise we say the series diverges. Sometimes we also write $\sum a_n$ instead of $\sum_{n=1}^{\infty} a_n$.

The role of a series is to study if the terms of a given sequence can be summed up to be a finite number. For example, $\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.1 + 0.01 + 0.001 + ... = 0.11111.... = \frac{1}{9}$ is convergent.

The most elementary method to determine if a given series converges is based on its definition, that is, we study the convergence of the sequence of its partial sums $\{s_N\}$.

Example 1. *We are going to show that the series $\sum r^n$ converges, where $|r| < 1$ is a constant.*

$s_N = \sum\limits_{n=1}^{N} r^n$, *so* $rs_N = r\sum\limits_{n=1}^{N} r^n = \sum\limits_{n=1}^{N} r^{n+1} = \sum\limits_{2}^{N+1} r^n$.

Subtracting these two equations, we obtain

$$(1-r)s_N = r - r^{N+1}$$

so

$$s_N = \frac{r - r^{N+1}}{1-r}$$

Taking the limit,

$$\sum r^n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{r - r^{N+1}}{1-r} = \frac{r}{1-r}$$

so the series converges.

Proposition 2. *If $\sum a_n$ and $\sum b_n$ are two convergent series and c is a constant, then:*

1. $\sum a_n + b_n = \sum a_n + \sum b_n$
2. $\sum a_n - b_n = \sum a_n - \sum b_n$
3. $\sum ca_n = c \sum a_n$

Example 3. *A series of the form $\sum ar^n$, where a and r are constant, is called a geometric series. When $|r| < 1$, we have shown that $\sum r^n$ converges to $\frac{r}{1-r}$, so $\sum ar^n = \frac{ar}{1-r}$ is also convergent.*

Example 4. $\sum 2^n 3^{1-n} = \sum 3 \times (\frac{2}{3})^n = 3 \times \frac{\frac{2}{3}}{1-\frac{2}{3}} = 6$

Proposition 5. *If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.*

The idea of proof is that $a_n = s_n - s_{n-1}$, and if $\{s_n\}$ converges, the difference $s_n - s_{n-1}$ will approach 0.

Example 6. *For $|r| \geq 1$, the series $\sum r^n$ diverges since the sequence $\{r^n\}$ doesn't converge to 0.*

The following examples shows the inverse of the previous proposition in general is not true, that is, $\lim\limits_{n\to\infty} a_n = 0$ does not necessarily imply $\sum a_n$ converges:

Example 7. *Consider the series $\sum \frac{1}{n}$.*

$$\begin{aligned}
S_{2^N} &= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + ... + (\frac{1}{2^{N-1}+1} + ... + \frac{1}{2^N}) \\
&> 1 + \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + ... + 2^{N-1} \times \frac{1}{2^N} \\
&= 1 + \frac{1}{2} + ... + \frac{1}{2} \\
&= 1 + \frac{N}{2}
\end{aligned}$$

We see $\lim\limits_{N\to\infty} S_{2^N} = +\infty$, so $\{S_N\}$ diverges, hence $\sum \frac{1}{n}$ diverges.

Example 8. *The series* $\sum \frac{1}{n^2+n}$ *converges:*
$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$*. So*

$$s_N = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + ... + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

$$\sum \frac{1}{n^2+n} = \lim_{N\to\infty} s_N 1 - \frac{1}{n+1} = 1$$

An application of series is to write a given decimal number into a fraction of integers.

Example 9. *Write* $1.3\overline{25} = 1.325252525...$ *as a fraction of integers.*

$$\begin{aligned} 1.3\overline{25} &= 1 + \frac{3}{10} + \frac{25}{10^3} + \frac{25}{10^5} + \frac{25}{10^7} + ... \\ &= 1 + \frac{3}{10} + \frac{25}{10} \sum (\frac{1}{100})^n \\ &= 1 + \frac{3}{10} + \frac{5}{2} \times \frac{\frac{1}{100}}{1 - \frac{1}{100}} \\ &= 1 + \frac{3}{10} + \frac{5}{2} \times \frac{1}{99} \\ &= \frac{656}{495} \end{aligned}$$

2 Convergence Tests of Series

In this section we are going to introduce some tests for the convergence of series. The proofs are omitted in this notes, and you may refer to the textbook for the proofs.

Proposition 10 (Integral Test)**.** $f(x)$ *is a positive and decreasing function on* $[K, +\infty)$ *for some constant* K*.* $\sum a_n$ *is a series such that* $a_n = f(n)$ *for any positive integer* $n \geq K$*. Then the series* $\sum a_n$ *converges if and only if the improper integral* $\int_K^{+\infty} f(x)\,dx$ *converges.*

Example 11. *We know the improper integral* $\int_1^{+\infty} \frac{1}{x^p}\,dx$ *converges for* $p > 1$*, diverges for* $0 < p \leq 1$*, so by Integtal Test. the series* $\sum \frac{1}{n^p}$ *converges for* $p > 1$ *and diverges for* $0 < p \leq 1$*.*

Example 12. *Determine if the series* $\sum \frac{\ln n}{n}$ *is convergent or divergent.*

Let $f(x) = \frac{\ln x}{x}$*. Note that* $f'(x) = \frac{1-\ln x}{x^2}$*, so* $f(x)$ *is positive and decreasing on* $[e, +\infty)$*.*

$\int_e^t \frac{\ln x}{x}\, dx = \int_e^t \ln x \, d \ln x = \frac{(\ln x)^2}{2}\Big|_e^t = \frac{(\ln t)^2 - 1}{2}$*, so*

$$\int_e^{+\infty} \frac{\ln x}{x}\, dx = \lim_{t\to+\infty} \int_e^t \frac{\ln x}{x}\, dx = \lim_{t\to+\infty} \frac{(\ln t)^2 - 1}{2} = +\infty$$

The improper integral diverges, so the series $\sum \frac{\ln n}{n}$ *diverges.*

Proposition 13 (Comparison Test)**.** $\sum a_n$ *and* $\sum b_n$ *are series. If there exists* $K > 0$ *such that for all* $n \geq K$*,* $0 < a_n < b_n$*, then:*

1. $\sum a_n$ *diverges implies* $\sum b_n$ *diverges*
2. $\sum b_n$ *converges implies* $\sum a_n$ *converges*

Example 14. *Consider the series* $\sum \frac{2}{n^2+3n}$*. Since* $0 < \frac{2}{n^2+3n} < \frac{2}{n^2}$*, and* $\sum \frac{2}{n^2}$ *converges, we conclude by Comparison Test that* $\sum \frac{2}{n^2+3n}$ *converges.*

Example 15. *We can also use Comparison Test to show* $\sum \frac{\ln n}{n}$ *diverges:*
$\frac{\ln n}{n} > \frac{1}{n} > 0$ *for all* $n > e$*, and* $\sum \frac{1}{n}$ *diverges, we conclude* $\sum \frac{\ln n}{n}$ *diverges.*

Proposition 16 (Limit Comparison Test)**.** $\sum a_n$ *and* $\sum b_n$ *are series with positive terms for all large* n*. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

Then both series converge or both series diverge.

Example 17. *Consider* $\sum \frac{1}{2^n-1}$*. Since*

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1$$

and $\sum \frac{1}{2^n}$ *converges, by the Limit Comparison Test, we get* $\sum \frac{1}{2^n-1}$ *converges.*

A sequence is called **alternating** if its terms are alternately positive and negative.

Proposition 18 (Alternating Series Test)**.** *If $\{a_n\}$ is a positive decreasing sequence or a negative increasing sequence, and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum(-1)^n a_n$ converges.*

Example 19. *The sequence $\{\frac{1}{n}\}$ is positive and decreasing, with $\lim_{n\to\infty} \dfrac{1}{n} = 0$, so by the Alternating Series Test, we conclude the alternating series $\sum \frac{(-1)^n}{n}$ converges.*

Example 20. *Determine if the series $\{(-1)^{n+1}\frac{n}{n^2+1}\}$ converges.*

Observe that the series is alternating. We will show the positive sequence $\{\frac{n}{n^2+1}\}$ is decreasing:

Let $f(x) = \frac{x}{x^2+1}$, then $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$, so the sequence $\{\frac{n}{n^2+1}\}$ is decreasing, and also

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0$$

So the Alternating Series Test implies the series $\sum(-1)^{n+1}\frac{n}{n^2+1}$ converges.

A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges. A series $\sum a_n$ is said to be conditionally convergent if $\sum |a_n|$ diverges but $\sum a_n$ converges.

Proposition 21 (Absolute Convergence Test)**.** *An absolutely convergent series converges.*

Example 22. *The series $\sum \frac{\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}}{n^2}$ converges absolutely since*

$$\sum \left|\frac{\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}}{n^2}\right| = \sum \frac{1}{n^2}$$

By Absolute Convergence Test, we conclude the series converges.

Example 23. *Consider $\sum \frac{\sin n}{n^2}$. $< |\frac{\cos n}{n^2}| \leq \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges, by Comparison Test, $\sum |\frac{\cos n}{n^2}|$ converges, so the Absolute Convergence Test implies $\sum \frac{\sin n}{n^2}$ converges.*

Proposition 24 (Ratio Test)**.** *$\sum a_n$ is a series:*

1. *If $\lim_{n\to\infty} |\dfrac{a_{n+1}}{a_n}| < 1$, then $\sum a_n$ converges absolutely.*

2. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$*, then* $\sum a_n$ *diverges.*

Example 25. *The series* $\sum (-1)^n \frac{n^3}{3^n}$ *converges absolutely:* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{1}{3}(\frac{n+1}{n})^3 = \frac{1}{3} < 1$*, so the Ratio Test implies the absolute convergence.*

Example 26. *The series* $\sum \frac{n!}{n^n}$ *converges absolutely:* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} (\frac{n}{n+1})^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$*, so the Ratio Test implies the absolute convergence.*

Proposition 27 (Root Test)**.** $\sum a_n$ *is a series:*

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$*, then* $\sum a_n$ *converges absolutely.*

2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$*, then* $\sum a_n$ *diverges.*

Example 28. $\sum (\frac{n+3}{2n+5})^n$*.* $\lim_{n\to\infty} \sqrt[n]{|(\frac{n+3}{2n+5})^n|} = \lim_{n\to\infty} \frac{n+3}{2n+5} = \frac{1}{2} < 1$*, so by the Root Test, the series converges absolutely.*