Sequences

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Definition 1. A **sequence** $\{a_n\}_{n=1}^{\infty}$ (often denoted as $\{a_n\}$ for short) is a list of numbers labelled by natural numbers: $a_1, a_2, a_3, ...$

A common way to describe the terms in a sequence $\{a_n\}$ is to give a formula involving n, called the defining equation. For example, if the sequence is given by $\{a_n\}$ with $a_n = \frac{n+1}{n}$, then the first five terms of the sequence are $a_1 = \frac{2}{1}$, $a_2 = \frac{3}{2}$, $a_3 = \frac{4}{3}$, $a_4 = \frac{5}{4}$, $a_5 = \frac{6}{5}$.

Sometimes it is very hard to find an explicit defining equation. Instead, we use some recursive formula to define a sequence. For example, the **Fibonacci sequence** $\{F_n\}$ is defined by $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. So the first few terms of this sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, ...$$

This is a sequence of great interest to mathematicians. For more story about it, the reader may refer to the WikiPedia page `https://en.wikipedia.org/wiki/Fibonacci_number`

Definition 2. A sequence $\{a_n\}$ has limit L if for any $\epsilon > 0$, there exists a positive integer N such that $n > N$ implies $|a_n - L| < \epsilon$.

If L is the limit of $\{a_n\}$, we say $\{a_n\}$ converges to L, and denote by

$$\lim_{n\to\infty} a_n = L$$

If $\{a_n\}$ has no limit, we say $\{a_n\}$ diverges.

If for any positive number M, there is positive integer N such that $n > N$ implies $a_n > M$, we say the sequence diverges to $+\infty$, and denote by

$$\lim_{n\to\infty} a_n = +\infty$$

If for any negative number M, there is positive integer N such that $n > N$ implies $a_n < M$, we say the sequence diverges to $-\infty$, and denote by

$$\lim_{n\to\infty} a_n = -\infty$$

Remark 3. The definition of the limit of a sequence intuitively says that $\{a_n\}$ has limit L if a_n can be arbitrarily close to L for sufficient large n.

Example 4. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} (-1)^n$ *diverges,* $\lim\limits_{n\to\infty} n = +\infty$

The following theorem tells us that we may make use of our knowledge about limit of functions to study limits of sequences:

Theorem 5. *If* $\lim\limits_{x\to+\infty} f(x) = L$*, then* $\lim\limits_{n\to\infty} f(n) = L$*. Similar results hold for the case of diverging to* $\pm\infty$*.*

Example 6. *Find* $\lim\limits_{n\to\infty} \frac{\ln n}{n}$.

We know by the L'Hospital's Rule that $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$*, so by the theorem,* $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

The limit of sequences follow some rules that are similar to those for the limits of functions:

Proposition 7. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences with* $\lim\limits_{n\to\infty} a_n = L_1$ *and* $\lim\limits_{n\to\infty} b_n = L_2$*, then:*

1. $\lim\limits_{n\to\infty} a_n \pm b_n = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L_1 \pm L_2$
2. $\lim\limits_{n\to\infty} \lambda a_n = \lambda(\lim\limits_{n\to\infty} a_n) = \lambda \lim\limits_{n\to\infty} a_n$ *for any real number* λ
3. $\lim\limits_{n\to\infty} a_n b_n = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n) = L_1 L_2$
4. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L_1}{L_2}$ *(*$L_2 \neq 0$*)*
5. $\lim\limits_{n\to\infty} a_n^p = (\lim\limits_{n\to\infty} a_n)^p = L_1^p$ *if* $p > 0$ *and* $a_n > 0$

Example 8. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \frac{1}{n}} = 1$

Proposition 9 (Squeeze Theorem/Sandwich Theorem)**.** *If $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences and there exists natural number N such that for any $n \geq N$, $a_n \leq b_n \leq c_n$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Corollary 10. *$\{a_n\}$ is a sequence. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. $-|a_n| \leq a_n \leq |a_n|$, and $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} -|a_n| = 0$. Applying the Squeeze Theorem, we conclude $\lim_{n\to\infty} a_n = 0$. □

Example 11. $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ *since* $\lim_{n\to\infty} |\frac{(-1)^n}{n}| = \lim_{n\to\infty} \frac{1}{n} = 0.$

Theorem 12. *If $f(x)$ is a function that is continuous at L, and $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = L$*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

Corollary 13. *If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.*

Example 14. $\lim_{n\to\infty} \cos(\frac{1}{n}) = \cos(\lim_{n\to\infty} \frac{1}{n}) = \cos 0 = 1$

Example 15. *r is a constant.*

If $r > 0$, we know

$$\lim_{x\to+\infty} r^x = \begin{cases} 0, & \textit{if } 0 \leq r \leq 1 \\ 1, & \textit{if } r = 1 \\ +\infty, & \textit{if } r > 1 \end{cases}$$

So we get

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \textit{if } 0 \leq r \leq 1 \\ 1, & \textit{if } r = 1 \\ +\infty, & \textit{if } r > 1 \end{cases}$$

When $r = 0$, $\lim_{n\to\infty} r^n = \lim_{n\to\infty} r^n 0 = 0$
When $-1 < r < 0$, $\lim_{n\to\infty} |r^n| = 0$, so $\lim_{n\to\infty} r^n = 0$.
When $r = -1$, the sequence becomes $\{(-1)^n\}$, which diverges.
When $r < -1$, $\{|r^n|\}$ diverges, so $\{r^n\}$ diverges.

Definition 16. $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all n. $\{a_n\}$ is **decreasing** if $a_n > a_{n+1}$ for all n. $\{a_n\}$ is **monotinoc** if it is increasing or decreasing.

$\{a_n\}$ is **bounded above** if there exists real number M such that $a_n < M$ for all n; $\{a_n\}$ is **bounded below** if there exists real number M such that $a_n > M$ for all n. $\{a_n\}$ is **bounded** if it is both bounded above and bounded below.

Theorem 17. *A bounded above increasing sequence converges; A bounded below decreasing sequence converges.*

Example 18. *The sequence* $\{1 + \frac{1}{n^3}\}$ *converges since it is decreasing and bounded below by* 1.