Polar Coordinates

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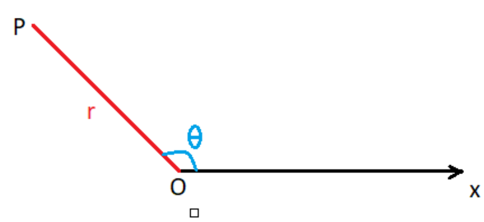
1 Polar Coordinates

There are different ways to locate a point on a plane, among which is the Cartesian coordinates that we have been using for a long time. Besides the Cartesian coordinates, another popular coordinate system for the plane is the **polar coordinates**, where the location of a point is described by its distance and direction to the origin.

P is a point on the plane. We connect the origin O and P by a line segment. The length of the line segment $\overline{OP}$ is denoted by r, and the angle whose terminal edge (recall that by default, the initial edge for an angle coincides with the positive x-axis) coincides with $\overline{OP}$ is denoted by θ (Note that the choice of θ is not unique, since angles differed by multiples of 2π will have terminal edge coincide). Then in polar coordinates, the point P is represented by the pair (r, θ).



A special convention is that given (r, θ), the polar coordinates of a point P, if $r < 0$, this denotes the point with polar coordinates $(-r, \theta + \pi)$. For example, the point $(-1, \frac{\pi}{2})$ in polar coordinates is the same as $(1, \frac{3\pi}{2})$.

If a point on the plane P has Cartesian coordinates (x, y) and polar coordinates (r, θ), by Trigonometry, we can obtain the conversion formulas between the two coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Example 1. *If a point has polar coordinates* $r = 3, \theta = -\frac{\pi}{6}$*, then its Cartesian coordinates can be recovered by* $x = 3\cos(-\frac{\pi}{6}) = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$*,* $y = 3\sin(-\frac{\pi}{6}) = 3 \times (-\frac{1}{2}) = -\frac{3}{2}$*, so its Cartesian Coordinates is* $(\frac{3\sqrt{3}}{2}, -\frac{3}{2})$*.*

We can also convert the other way round:

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan\theta = \frac{y}{x} \text{ if } x \neq 0 \end{cases}$$

You will need to determine the value for θ after you find the value for $\tan\theta$. If we use the inverse tangent function, we can choose θ in the following way:

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x}, \text{ if } (x, y) \text{ is in Quadrant I or IV} \\ \pi + \tan^{-1}\frac{y}{x}, \text{ if } (x, y) \text{ is in Quadrant II or III} \end{cases}$$

Example 2. *If* P *is a point with Cartesian coordinates* $(-1, -1)$*, find the polar coordinates for* P*.*

$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$*.* $\tan\theta = \frac{-1}{-1} = 1$*, and* P *is in the Quadrant III, so* $\theta = \pi + \tan^{-1}(1) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$*. In polar coordinates the point* P *is* $r = \sqrt{2}, \theta = \frac{5\pi}{4}$*.*

When the point has x-coordinate to be 0, it lies on the y-axis, so its angle is $\theta = \frac{\pi}{2}$ if it's on positive y-axis, and $\theta = -\frac{\pi}{2}$ if it is on the negative y-axis.

Example 3. *The polar coordinates for the point* $(0, 3)$ *is* $r = 3, \theta = \frac{\pi}{2}$*, and the polar coordinates for the point* $(0, -3)$ *is* $r = 3, \theta = -\frac{\pi}{2}$

Similar to Cartesian coordinates, the polar coordinates can be used to describe not only points, but also curves, an equation $F(r, \theta) = 0$ denotes the set of all points on the plane whose polar coordinates satisfy this equation.

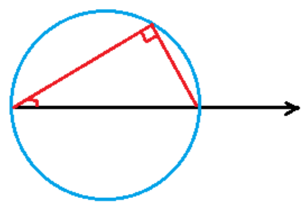
Example 4. *The circle of radius* R *with centre at origin corresponds to the equation* $r = R$*, since it is the set of points whose distance to the origin is* R*.*

A ray coming out from Origin can be described by the equation $\theta = \theta_0$*, where* θ_0 *is the angle formed by the positive* x*-axis and this ray. For example, the equation for the diagonal ray of the Quadrant II is* $\theta = \frac{3\pi}{4}$*.*

Example 5. *What is the polar equation of the circle of radius* 2 *with centre at* $(1,0)$*?*

For angle θ*, the corresponding radius is* $2\cos\theta$*, so the equation is*

$$r = 2\cos\theta$$



You can also fine some other interesting examples in the textbook, for example, $r = 1 + \sin\theta$ (cardioid) and $r = \cos 2\theta$ (four-leaved rose).

Example 6. *Find the intersections of the four leaved rose* $r = 2\cos\theta$ *and the circle* $r = \frac{1}{2}$

We need to first find points of intersection of the two curves. They intersect at the points on the four leaved rose such that $r = \pm\frac{1}{2}$*, i.e.,* $|r| = \frac{1}{2}$*. We get the equation*

$$\frac{1}{2} = |r| = |\cos 2\theta|$$

and the solutions are $\pm\frac{\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{6}$*, which correspond to the points in Cartesian coordinates:*

$$(\pm\frac{\sqrt{3}}{4}, \pm\frac{1}{4}), (\pm\frac{1}{4}, \pm\frac{\sqrt{3}}{4})$$

2 Calculus for Polar Coordinates

Given the polar equation $r = f(\theta)$ of a curve, we can find the slope of the curve by the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d(r\sin\theta)}{d\theta}}{\frac{d(r\cos\theta)}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

In particular, if the curve passes through the origin, its slope is given by

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta}{\frac{dr}{d\theta}\cos\theta} = \tan\theta$$

when $\frac{dr}{d\theta} \neq 0$, and this formula agrees with our geometric understanding of the slope.

Example 7. *Find the points on the curve $r = 1 + \sin\theta$ at which the tangent line is horizontal or vertical.*

We know the tangent line is horizontal when the slope is zero; the tangent line is vertical when the slope tends to infinity. So we compute the slope first.

$$\begin{aligned}
\frac{dy}{dx} =& \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
=& \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} \\
=& \frac{(1+2\sin\theta)\cos\theta}{(1-\sin^2\theta) - \sin^2\theta - \sin\theta} \\
=& \frac{(1+2\sin\theta)\cos\theta}{-2\sin^2\theta - \sin\theta + 1}
\end{aligned}$$

The numerator is zero when $(1+2\sin\theta)\cos\theta = 0$, i.e., $\sin\theta = -\frac{1}{2}$ or $\cos\theta = 0$, we get $\theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{2}, -\frac{\pi}{2}$.

The denominator is zero when $-2\sin^2\theta - \sin\theta + 1 = 0$, i.e., $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$, we get $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$.

Note $-\frac{\pi}{2}$ is a solution to both the numerator and denominator, and

$$\lim_{\theta\to-\frac{\pi}{2}} \left|\frac{dy}{dx}\right| = \lim_{\theta\to-\frac{\pi}{2}} \left|\frac{(1+2\sin\theta)\cos\theta}{-2\sin^2\theta - \sin\theta + 1}\right| = \infty$$

So the tangent line is vertical at this point.

We conclude the tangent line is horizontal when $\theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{2}$, they correspond to points $(\frac{\sqrt{3}}{4}, -\frac{1}{4})$, $(-\frac{\sqrt{3}}{4}, -\frac{1}{4})$, $(0, 2)$ in Cartesian coordinates.

The tangent line is vertical when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$, they correspond to points $(\frac{3\sqrt{3}}{4}, \frac{3}{4})$, $(-\frac{3\sqrt{3}}{4}, \frac{3}{4})$, $(0, 0)$ in Cartesian coordinates.

We can compute the arc length of a polar curve of the form $r = f(\theta)$, with $a \le \theta \le b$. This in in Cartesian coordinates a parametric curve

$$(r\cos\theta, r\sin\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$$

Theorem 8. *The arc length of the parametric curve $r = f(\theta)$, with $a \le \theta \le b$ is*

$$\int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

Proof.

$$\begin{aligned}
&\int_a^b \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta \\
=&\int_a^b \sqrt{(\frac{r\cos\theta}{d\theta})^2 + (\frac{r\sin\theta}{d\theta})^2}\, d\theta \\
=&\int_a^b \sqrt{(\frac{dr}{d\theta}\cos\theta - r\sin\theta)^2 + (\frac{dr}{d\theta}\sin\theta + r\cos\theta)^2}\, d\theta \\
=&\int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta
\end{aligned}$$

□

Example 9. *Compute the arclength of the four cadioiod $r = 1 + \sin\theta$.*

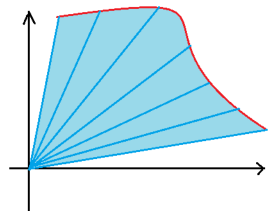
$$\int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta = 8$$

We can also use polar coordinates to compute some area.

Theorem 10. *A polar curve is given by the equation $r = f(\theta)$, with $a \le \theta \le b$. Connect the two endpoints to the origin, then the area enclosed can be computed by*

$$\int_a^b \frac{r^2}{2}\, d\theta$$

Proof. We can divide $[a, b]$ into n pieces of length $\Delta\theta = \frac{b-a}{n}$, with endpoints $a = \theta_0 < \theta_1 < ... < \theta_{n-1} < \theta_n = b$. Choose a point $\theta_i^* \in [\theta_{i-1}, \theta_i]$. The rays $\theta = \theta_1, ..., \theta = \theta_{n-1}$ divide the region into n parts, and when n is big, each of



them can be approximated by a sector of a circle with radius $f(\theta_i^*)$ and angle $\Delta\theta$, whose are is $\frac{r^2\theta}{2}$. So taking the limit of the Riemann sum, we obtain the area enclosed is

$$\lim_{\Delta\theta\to 0}\sum_{i=1}^{n}\frac{f(\theta_i^*)^2}{2}\Delta\theta = \int_a^b \frac{f(\theta)^2}{2}\,d\theta = \int_a^b \frac{r^2}{2}\,d\theta$$

□

Example 11. *Find the region enclosed by one loop of the four leaved rose $r = \cos 2\theta$.*

$$\int_0^{2\pi}\frac{(\cos 2\theta)^2}{2}\,d\theta = \int_0^{2\pi}\frac{1+\cos 4\theta}{4}\,d\theta = \int_0^{8\pi}\frac{1+\cos u}{16}\,du = \frac{\pi}{2}$$

Example 12. *Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1+\sin\theta$.*

We first find the points of intersection of the two curves:

$$\begin{cases} r = 3\sin\theta \\ r = 1+\sin\theta \end{cases}$$

We get $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$, and $r = \frac{3}{2}$. So the area is

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\frac{(3\sin\theta)^2}{2} - \frac{(1+\sin\theta)^2}{2}\,d\theta = \pi$$