Applications to Physics

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We are going to use the language of integration to describe some phenomena in physics. We will concentrate on the case of 1-dimensional physics.

1 Force and Work

In physics, the **work** done by a constant force F with distance d along the direction of the force is defined to be

$$W = F \times d$$

i.e., Work = Force × Distance. And the unit for work is Joule (denoted by J). 1 J = 1 N × 1 m.

But in reality, force is not always constant, but a function in terms of position, say $F = f(x)$. In such circumference, we can also define the work using integration: Divide the interval $[a, b]$ into n pieces of length $\Delta x = \frac{b-a}{n}$, and in each interval, we take a point $x_i^* \in [x_{i-1}, x_i]$. When Δx is small, the force is almost constantly equal to $f(x_i^*)$, so an approximation of the actual work is:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Taking the limit as $\Delta x \to 0$, we obtain the actual work done by the force:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, dx$$

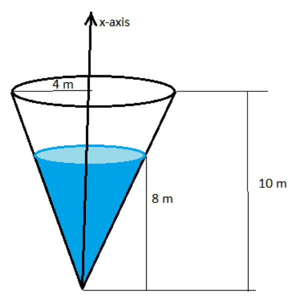
Example 1. ***Hooke's Law*** *states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x (for small x): $f(x) = kx$, where k is a positive constant.*

A force of $40N$ *is required to hold a sping that has been stretched from its natural length of* 10 *cm to a length of* 15 *cm. How much work is done in stretching the spring from* 15 *cm to* 18 *cm?*

First we figure out k: $40 = f(0.15-0.1) = k(0.05) = k\times 0.05$, *so* $k = 800$. *The work done in stretching the spring from* 15 *to* 18 *is*

$$\int_{0.15-0.1}^{0.18-0.1} f(x)\,dx = \int_{0.05}^{0.08} 800x\,dx = 1.56$$

Example 2. *A tank has the shape of an inverted circular cone with height 10 meters and base radius* 4 *meters. It is filled with water to a height of* 8 *meters. The density of water is* 1000 *kilograms per cubic meters. Find the work required to empty the tank by pumping all the water to the top of the tank.*



We build a x*-axis vertically upward with origin at the vertex of the cone. We divide the interval* $[0, 8]$ *into* n *pieces of length* $\Delta x = \frac{8}{n}$. *The volume for the piece between* $[x_{i-1}, x_i]$ *can be approximated by a cylinder with base radius* $\frac{2}{5}x_i$ *and height* Δx. *So the work done on this piece is approximated by*

$$\pi(\frac{2}{5}x_i)^2 \times 1000 \times 9.8 \times (10-x)$$

So the actual work can be achieved by taking the limit of the Riemann sum:

$$\lim_{\Delta x\to 0}\sum_{i=1}^{n}\pi(\frac{2}{5}x_i)^2\Delta x\times 1000\times 9.8\times(10-x) = \int_0^8 1568x^2(10-x)\,dx = \frac{3211264}{3}\pi \approx 3.4\times 10^6$$

The actual work is about 3.4×10^6 *Joules.*

2 Hydrostatic Pressure

The **pressure** at depth d under the surface of liquid is defined to be

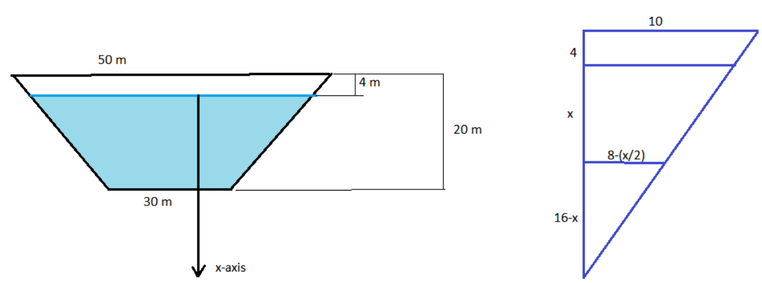
$$P = \rho g d$$

where ρ is the density of liquid and g is the gravitational acceleration. The pressure at depth d represents the hydrostatic force a plate at this level feels per unit area. So

$$F = P \times A$$

where A is the area of a plate and F is the hydrostatic force.

Example 3. *A dam of trapezium shape has upper width* 50 *meters, lower width* 30 *meters and height* 20 *meters. Find the force on the dam due to hydrostatic pressure if the water level is* 4 *meters from the top of the dam.*



We choose x-axis pointing downward and origin at the water level. Then at x, the width of the dam is $30 + 2 \times (8 - \frac{x}{2}) = 46 - x$.

Divide $[0, 16]$ into pieces of length $\Delta x = \frac{16}{n}$ and choose $x_i^* \in [x_{i-1}, x_i]$. When Δx is small, the pressure on the stripe corresponding to the interval $[x_{i-1}, x_i]$ is almost constantly equal to

$$\rho g x_i^*$$

So the hydrostatic force on this stripe is almost

$$(\rho g x_i^*)(46 - x_i^*)\Delta x$$

Taking the limit of the Riemann sum, the total hydrostatic force is

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} (\rho g x_i^*)(46 - x_i^*)\Delta x = \int_0^{16} \rho g x(46 - x)\, dx \approx 4.43 \times 10^7$$

(in the above computation we take $\rho = 1000$ and $g = 9.8$)

3 Moments and Centre of Mass

We first deal with the discrete case. Build a coordinate system on a plane. If there are n particles on a line at $(x_1, y_1), ..., (x_n, y_n)$, with mass $m_1, ..., m_n$ respectively, we define the **moment of the system** about the y-axis to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system** about the x-axis to be

$$M_x = \sum_{i=1}^{n} m_i y_i$$

We define the **centre of mass** of the system to be the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m} = \frac{\sum_{i=1}^{n} m_i x_i}{m}, \bar{y} = \frac{M_x}{m} = \frac{\sum_{i=1}^{n} m_i y_i}{m}$$

where $m = \sum_{i=1}^{n} m_i$.

Example 4. *Find the moment and centre of mass of the system of particles that have mass* $3, 4, 8$ *at* $(-1, 1), (2, -1), (3, 2)$ *respectively.*

$$M_y = (-1) \times 3 + 2 \times 4 + 3 \times 8 = 29, M_x = 1 \times 3 + (-1) \times 4 + 2 \times 8 = 15$$

$$\bar{x} = \frac{M_y}{3} = \frac{29}{15}, \bar{y} = \frac{M_x}{15} = 1$$

Next we will study the centre of mass (also called **centroid**) for a plate with uniform density. The **Symmetric Principle** in physics says for a plate of uniform mass, the centre of mass is on its symmetric axis, so for a rectangular shape plate, its centre of mass is at its geometric centre.

Now consider a plate that is bounded between $y = f(x)$ and x-axis on the interval $[a, b]$. We use the rectangles for Midpoint Approximation to approximate this region. For the i-th rectangle, its centre is at $(x_i^*, \frac{f(x_i^*)}{2})$, and its mass is $\rho f(x_i^*)\Delta x$, where $x_* = \frac{x_{i-1}+x_i}{2}$. So the moment of the system of rectangles is

$$\begin{cases} M_y = \sum_{i=1}^{n} (\rho f(x_i^*)\Delta x) x_i^* \\ M_x = (\rho f(x_i^*)\Delta x) \frac{f(x_i^*)}{2} \end{cases}$$

Taking the limit as $\Delta x \to 0$, we get the moment of the plate:

$$\begin{cases} M_y = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (\rho f(x_i^*)\Delta x) x_i^* = \rho \int_a^b x f(x)\, dx \\ M_x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (\rho f(x_i^*)\Delta x) \frac{f(x_i^*)}{2} = \rho \int_a^b \frac{f^2(x)}{2}\, dx \end{cases}$$

And divided by the mass of the plate, which is $\rho \int_a^b f(x)\, dx$, we get the centre of mass of the plate:

$$\left(\frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}, \frac{\int_a^b \frac{f^2(x)}{2}\, dx}{\int_a^b f(x)\, dx}\right)$$

Remark 5. We see in the above expression that for a plate of uniform density, its centre of mass is independent of the density.

Example 6. *Find the centre of mass of a semicircular plate of radius r.*

We put the plate in the coordinate system so that the circular boundary is the graph of the function $y = \sqrt{r^2 - x^2}$, $-1 \le x \le 1$.

The centre of mass is

$$\left(\frac{\int_{-1}^1 x\sqrt{r^2 - x^2}\, dx}{\frac{\pi r^2}{2}}, \frac{\int_{-1}^1 (r^2 - x^2)\, dx}{\frac{\pi r^2}{2}}\right) = (0, \frac{4r}{3\pi})$$

More generally, if $y = f(x)$ and $y = g(x)$ are functions on $[a, b]$ and $f(x) \ge g(x)$ on $[a, b]$, then the centre of mass for the plate enclosed by $y = f(x)$, $y = g(x)$, $x = a$, $x = b$ is

$$\left(\frac{\int_a^b x(f(x) - g(x))\, dx}{\int_a^b f(x) - g(x)\, dx}, \frac{\int_a^b \frac{f^2(x) - g^2(x)}{2}\, dx}{\int_a^b f(x) - g(x)\, dx}\right)$$

Another useful proposition about the centre of mass is the following, which describes the relation of the centre of mass of a plate and the centres of mass of its parts:

Proposition 7. *A plate region D is of area A with centre of mass at (x_0, y_0), and this region is union of smaller regions $D_1, ..., D_n$ that intersect with each other only possibly along boundary. If the area for D_i is A_i and the centre of mass for D_i is (x_i, y_i), then:*

$$x_0 = \frac{\sum_{i=1}^n m_i x_i}{m}, y_0 = \frac{\sum_{i=1}^n m_i y_i}{m}$$

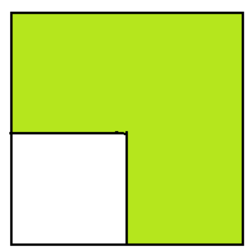
Example 8. *A square of edge length* 1 *is removed from a square of edge length* 2 *as shown in the figure below. Find the centre of mass of the remaining part.*

If we build the coordinate system with origin at the lower left corner of the square, then by the Symmetric Principle, the centre of mass of the big rectangle is at $(1, 1)$ *and the centre of mass of the small rectangle is* $(\frac{1}{2}, \frac{1}{2})$.

By the above Proposition, if we assume the centre of mass of the remaining part is (x_0, y_0)*, then:*

$$\begin{cases} 1 \times \frac{1}{2} + 3 \times x_0 = 4 \times 1 \\ 1 \times \frac{1}{2} + 3 \times y_0 = 4 \times 1 \end{cases}$$

We obtain $x_0 = \frac{7}{6}, y_0 = \frac{7}{6}$*. So the centre of mass for the remaining part is at* $(\frac{7}{6}, \frac{7}{6})$.



Proposition 9 (Pappus Theorem)**.** *R is a region on the plane that lies entirely on one side of a line l. Then the volume of the solid obtained by rotating R around l equals the product of the area of R and the distance d travelled by the centre of mass of R.*

Example 10. *A torus is formed by rotating a circle of radius r around a line in the plane of the circle with distance R to the centre of the circle. By the Pappus Theorem, the volume of the torus is*

$$(\pi r^2) \times 2\pi R = 2\pi^2 r^2 R$$