Parametric Curves

Liming Pang

We have seen that one way to represent a curve on a plane is by writing down an equation that tells us the relation between the x and y coordinates that points on the curve need to satisfy. For example, we can use $y = f(x)$ to represent the graph of the function $f$, and $x^2 + y^2 = 1$ to represent the unit circle.

At the same time, we can view a given curve as the trajectory of a moving particle. If a particle moves on the plane, then its location $(x, y)$ is a function of time $t$, i.e., $x = x(t)$ and $y = y(t)$. We can thus represent the trajectory of the particle by $(x, y) = (x(t), y(t))$. A curve given in this way is called a parametric curve. In many circumstances, the variable $t$ is taken within some closed interval $[a, b]$. In such cases, we say the initial point of the curve is $(x(a), y(a))$ and the terminal point of the curve is $(x(b), y(b))$.

Example 1. If $y = f(x)$ is a function, then its graph can be parameterised by $(t, f(t))$.

Example 2. The parametric curve $(\cos t, \sin t)$, $t \in [0, 2\pi]$ describes the unit circle with both initial and terminal points at $(1, 0)$. The curve starts at $(1, 0)$, goes in counterclockwise direction and returns to $(1, 0)$.

The parametric curve $(\cos t, \sin t)$, $t \in [0, \pi]$ describes the upper half of the unit circle, starting from $(1, 0)$ and ending at $(-1, 0)$.

The parametric curve $(\cos t, \sin t)$, $t \in [0, 4\pi]$ also describes the unit circle as a curve, but this time it goes around the unit circle two times. So this one is regarded as a different parametric curve compared with the previous one.

The parametric curve $(\cos(-t), \sin(-t))$, $t \in [0, 2\pi]$ also describes the unit circle as a curve, but this time it goes clockwise. So this one is also regarded as a different parametric curve compared with the previous ones.
Example 3. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the trace of the parametric curve $(a \cos t, b \sin t), 0 \leq t \leq 2\pi$

Example 4. The cycloid is the trace of a point on a circle as the circle rolls along a straight line. Find parametric equations for the cycloid if the circle has radius $r$ and starts at origin rolling along positive $x$-axis, and the point $P$ is at bottom at the beginning.

We can let the circle travels in constant velocity $r$, so its centre will be at $(rt, r)$ at time $t$, and the angle formed between $P$ and the point tangent to the ground is $\frac{\pi}{2} = t$, so the parametric equation of the trace of $P$ is:

$$(rt - r \sin t, r - r \cos t) = (r(t - \sin t), r(1 - \cos t))$$

If we understand $t$ as the time in the parametric equation $(x(t), y(t))$ for the parametric curve, then its velocity at $t_0$, the rate of change of the displacement at this time, is given by

$$\lim_{t \to t_0} \left( \frac{x(t) - x(t_0)}{t - t_0}, \frac{y(t) - y(t_0)}{t - t_0} \right) = (x'(t_0), y'(t_0))$$
The vector \((x'(t_0), y'(t_0))\) is called the **tangent vector** of the parametric curve \((x(t), y(t))\) at \(t = t_0\).

**Example 5.** Consider the parametric curve \((\cos t, \sin t), t \in [0, 2\pi]\). Find the tangent vector of the curve at the point \((-1, 0)\).

The point \((-1, 0)\) corresponds to \(t = \pi\). The tangent vector at this point is

\[
\begin{align*}
((\cos t)', (\sin t)) \bigg|_{t=\pi} &= (-\sin \pi, \cos \pi) = (0, -1)
\end{align*}
\]

**Proposition 6.** If \((x(t), y(t))\) is a parametric curve, and \(x'(t_0) \neq 0\), then the slope of the curve at \((x(t_0), y(t_0))\) is \(\frac{y'(t_0)}{x'(t_0)}\).

**Proof.** By the Chain Rule, \(\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}\), so \(\frac{dy}{dx} = \frac{dy/dt}{dx/dt}\).

The slope is \(\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}\), so at \(t_0\) the slope is \(\frac{y'(t_0)}{x'(t_0)}\). \(\square\)

**Example 7.** Find the line tangent to the curve \(x = y^2\) at \((1, 1)\).

The curve can be parameterised as \((t^2, t)\). The point \((1, 1)\) corresponds to \(t = 1\). The tangent vector at \(t = 1\) is

\[
\begin{align*}
((t^2)', t') \bigg|_{t=1} &= (2t, 1) \bigg|_{t=1} = (2, 1)
\end{align*}
\]

This implies the tangent line of the curve passing through \((1, 1)\) is parallel to the vector \((2, 1)\), which means the slope of the tangent line is \(2 = \frac{2}{1} = 2\). So the equation of the tangent line is \(y - 1 = 2(x - 1)\).
We can also compute higher order derivatives:

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dt}{dx} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left( \frac{dy}{dt} \frac{dx}{dt} \right)}{\frac{dx}{dt}} = \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3}
\]

**Example 8.** Consider the circle given by the parametric equation \((\cos t, \sin t), 0 \leq t \leq 2\pi\).

\[
\frac{d^2 y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3}
\]

When \(0 < t < \pi\), \(\frac{d^2 y}{dx^2} < 0\), the corresponding curve (upper semicircle) is concave; when \(\pi < t < 2\pi\), \(\frac{d^2 y}{dx^2} > 0\), the corresponding curve (lower semicircle) is convex.

The idea of tangent vector motivates the following method for computing the arc length of a parametric curve:

**Theorem 9.** The arc length of the parametric curve \((x(t), y(t)), t \in [a, b]\) is

\[
\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt
\]

**Example 10.** Find the arc length of the parameterised curve \((t - \sin t, 1 - \cos t)\) on \(t \in [0, 2\pi]\)

\[
\int_0^{2\pi} \sqrt{[(t-\sin t)]^2 + [(1-\cos t)]^2} \, dt = \int_0^{2\pi} \sqrt{2(1-\cos t)} \, dt \\
= \int_0^{2\pi} \sqrt{2(2\sin^2 \frac{t}{2})} \, dt = \int_0^{2\pi} 2\sin \frac{t}{2} \, dt = 8
\]
Another application of tangent vectors is to compute the area bounded by the parametric curve \((x(t), y(t))\) and the \(x\)-axis for \(a \leq t \leq b\) when the curve coincide with the graph of a function \(y = f(x)\):

The area is

\[
\int_{x(a)}^{x(b)} f(x) \, dx = \int_{a}^{b} y(t) \, dx(t) = \int_{a}^{b} y(t)x'(t) \, dt
\]

**Example 11.** Find the area bounded by the parametric curve

\[(t - \sin t, 1 - \cos t)\]

and \(x\)-axis for \(0 \leq t \leq 2\pi\).

\[
\int_{0}^{2\pi} (1 - \cos t)(t - \sin t)' \, dt = \int_{0}^{2\pi} (1 - \cos^2 t)^2 \, dt = 3\pi
\]